



Cohomology Rings of Quasitoric Bundles

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Abstract. The classical Bernstein-Kushnirenko-Khovanskii theorem (or, the BKK theorem, for short) computes the intersection number of divisors on toric variety in terms of volumes of corresponding polytopes. In [PK92b], it was observed by Pukhlikov and the first author that the BKK theorem leads to a presentation of the cohomology ring of a toric variety as a quotient of a ring of differential operators with constant coefficients by the annihilator of an explicit polynomial.

In this paper we generalize this construction to the case of quasitoric bundles. These are fiber bundles with generalized quasitoric manifolds as fibers. First we obtain a generalization of the BKK theorem to this case. Then we use recently obtained descriptions of the graded-commutative algebras which satisfy Poincaré duality to give a description of cohomology rings of quasitoric bundles.

1. Introduction

In this paper we study cohomology rings of quasitoric bundles, a class of fiber bundles over a closed orientable smooth manifold with a generalized quasitoric manifold as a fiber. A generalized quasitoric manifold is a generalization of a toric variety to smooth category, hence the first examples of a quasitoric bundle are a nonsingular complete toric variety and a quasitoric manifold, both viewed as fiber bundles over a point.

Description of cohomology rings of toric varieties. Let us start with a brief recollection of the classical toric case. The cohomology ring of a smooth toric variety has several descriptions. In this paper we mostly focus on the following three descriptions: Stanley-Reisner, Brion's and Pukhlikov-Khovanskii (or, virtual polytope description).

Let $T \simeq (\mathbb{C}^*)^n$ be an algebraic torus with character lattice M (that is, the abelian group of (continuous) characters $\text{Hom}(T, S^1)$) and lattice of one-parameter subgroups N (that is, the lattice dual to M). Further, let X_Σ be a smooth projective toric variety given by a fan $\Sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Denote the set of rays of Σ by $\Sigma(1) = \{\rho_1, \dots, \rho_s\}$ and their primitive generators in N by e_1, \dots, e_s .

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Recall the following well-known description of the cohomology ring of X_Σ (see, for instance, [CLS11, Theorem 12.4.1]):

$$H^*(X_\Sigma, \mathbb{R}) \simeq \mathbb{R}[x_1, \dots, x_s]/(I + J) =: R_\Sigma,$$

where I is generated by monomials $x_{i_1} \cdots x_{i_t}$ such that $\rho_{i_1}, \dots, \rho_{i_t} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \langle \sum_{i=1}^s \chi(e_i)x_i : \chi \in M \rangle$. Note that I depicts the Stanley-Reisner ideal of the fan Σ and therefore we refer to this description as the *Stanley-Reisner description* of $H^*(X_\Sigma, \mathbb{R})$.

On the other hand, Brion used explicit localization theorem in equivariant cohomology to compute the cohomology ring of a smooth toric variety. Let M be the character lattice of the torus T and set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{R})$. Let Σ be a smooth projective fan in $N_{\mathbb{R}}$. A map $f: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is *piecewise polynomial* if for any $\sigma \in \Sigma$, the map $f|_{\sigma}: \sigma \rightarrow \mathbb{R}$ extends to a polynomial function on the linear space $\text{span}_{\mathbb{R}}\{\sigma\}$, i.e. a piecewise polynomial function f on Σ is a collection of compatible polynomial functions $f_{\sigma}: \sigma \rightarrow \mathbb{R}$.

Let us denote by PP_{Σ} the ring of piecewise polynomial functions on Σ with respect to pointwise addition and multiplication. Note that the character lattice M naturally injects in PP_{Σ} as global linear functions on $N_{\mathbb{R}}$. Brion shows that one has the following description of the cohomology ring of a toric variety:

$$H^*(X_\Sigma, \mathbb{R}) \simeq \text{PP}_{\Sigma}/\langle M \rangle,$$

where $\langle M \rangle$ is the ideal generated by $M \subset \text{PP}_{\Sigma}$.

Finally, given line bundles L_1, \dots, L_t on X_Σ , one can directly compute a top degree cup product $c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t}$ in $H^*(X_\Sigma, \mathbb{R})$ by using Bernstein-Kushnirenko-Khovanskii theorem (or, the BKK theorem, for short) [Kou76] (see also [Ber75, BKH76]). Here, $c_1(L_i) \in H^2(X_\Sigma, \mathbb{R})$ denotes the first Chern class of L_i . More precisely, as any line bundle on X_Σ is the difference of two ample line bundles, it suffices to evaluate products $c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t}$ with all L_i being ample. As is well-known in toric geometry, the ample line bundles L_i correspond to polytopes P_i whose normal fans coarsen the fan Σ .

Using the BKK Theorem, we obtain

$$c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t} = n! \cdot \underbrace{V(P_1, \dots, P_1, \dots, P_t, \dots, P_t)}_{\substack{k_1 \text{ times} \quad k_t \text{ times}}}$$

where $V(P_1, \dots, P_1, \dots, P_t, \dots, P_t)$ denotes the mixed volume of the n -tuple $(P_1, \dots, P_1, \dots, P_t, \dots, P_t)$. In [PK92b, Section 1.4], Pukhlikov and the first author observed that the information on these cup products suffices to regain a description of the cohomology ring $H^*(X_\Sigma, \mathbb{R})$. More precisely, they showed that

$$H^*(X_\Sigma, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_\Sigma)/\text{Ann}(\text{Vol}),$$

where $\text{Diff}(\mathcal{P}_\Sigma)$ is the ring of differential operators with constant coefficients on the space of virtual polytopes associated with Σ and $\text{Ann}(\text{Vol})$ is the ideal of differential operators which annihilates the volume polynomial. As virtual polytopes play a crucial role in this description, we refer to it as the *virtual polytope description* of the cohomology ring $H^*(X_\Sigma, \mathbb{R})$.

Cohomology rings of quasitoric bundles. Now let $T \simeq (S^1)^n$ be a compact torus. A *quasitoric manifold* is a smooth $2n$ -dimensional manifold with a smooth effective locally standard T -action such that the orbit space is diffeomorphic to a simple polytope P . This notion was introduced as a topological counterpart of a nonsingular projective toric variety (or, toric manifold) with the restricted action of $(S^1)^n \subset (\mathbb{C}^*)^n$ in the seminal paper [DJ91]. Quasitoric manifolds have been studied intensively since 1990s and have already found various applications in homotopy theory [CMS08, HKS16, HK17], unitary [BPR07, LW16] and special unitary bordism [LP16, LLP18], hyperbolic geometry [BP16, BEM17, BGLV20], and other areas of research.

Quasitoric manifolds acquire smooth structures as orbit spaces of moment-angle manifolds by freely acting tori of maximal possible rank [BP02, BPR07]. In Section 2, we define a *generalized quasitoric manifold* to be a partial quotient obtained as an orbit space of a moment-angle-complex over a starshaped sphere by a freely acting torus of the maximal possible rank. It is a locally standard torus manifold [Mas99, HM03, MP06] with a poset of characteristic submanifolds being isomorphic to a complete simplicial fan.

Similar to quasitoric manifolds, generalized quasitoric manifolds have a combinatorial description: an omnioriented generalized quasitoric manifold is uniquely determined by a pair (Σ, Λ) , where Σ is a complete simplicial fan and $\Lambda: \Sigma(1) \rightarrow N$ is a map such that $\Lambda(\rho_1), \dots, \Lambda(\rho_l)$ can be extended to a basis of N , whenever $\rho_1, \dots, \rho_l \in \Sigma(1)$ form a cone.

In particular, for a smooth complete fan $\Sigma \subset N_{\mathbb{R}}$ and $\Lambda(\rho) = e_{\rho}$, the primitive ray generator, the corresponding generalized quasitoric manifold is equivariantly diffeomorphic to a nonsingular complete toric variety X_{Σ} (considered with the restricted action of $(S^1)^n \subset (\mathbb{C}^*)^n$).

For a principal T -bundle $p: E \rightarrow B$ over a closed orientable real manifold B and a generalized quasitoric manifold $X_{\Sigma, \Lambda}$, let $E_{\Sigma, \Lambda} := (E \times X_{\Sigma, \Lambda}/T)$ be the corresponding *quasitoric bundle*; that is, a locally trivial fiber bundle with fiber $X_{\Sigma, \Lambda}$ associated with the toric bundle p (see Section 3 for details).

A particular example of quasitoric bundles is formed by *toric bundles*; that is, we consider associated bundles $E_{\Sigma} := (E \times X_{\Sigma}/T)$ coming from smooth complete toric varieties X_{Σ} . A generalization of the Stanley-Reisner type description for the cohomology ring $H^*(E_{\Sigma}, \mathbb{R})$ was obtained by Sankaran and Uma [SU03]. In [Hof19], it was noticed that Brion’s description also generalizes to the case of toric bundles. Finally, in [HKM20] a generalization of the Pukhlikov-Khovanskii description was obtained in the case of toric bundles with smooth projective fibers. Descriptions of cohomology rings obtained in [Hof19] and [HKM20] showed to be useful for computation of the ring of conditions of horospherical homogeneous spaces. Such a computation involves taking the direct limit of cohomology rings of toric bundles with the same base, for which the descriptions of [Hof19, HKM20] are more suitable. On the other side, in [DJ91] the Stanley-Reisner description was extended to the cohomology rings of quasitoric manifolds and in [DKU19] it was further extended to the case of quasitoric bundles. Moreover, in [AM16] the cohomology rings of quasitoric manifolds were described via volume polynomial on the space of analogous multi-polytopes.

In this paper, we complete this picture and obtain a generalization of all the three descriptions of the cohomology ring of a smooth toric variety to the case of quasitoric bundles. In particular, we extend the results of [HKM20] to the case of toric bundles with complete (but not necessarily projective) fibers and give an independent proof of the results of [DKU19] in the quasitoric bundle case.

Our main tool in the computation of cohomology rings is a topological version of the BKK theorem for quasitoric bundles which computes top-degree intersection numbers of cohomology classes of $E_{\Sigma, \Lambda}$ (Theorem 3.5).

More precisely, we reduce the computation of intersection numbers on the quasitoric bundle to intersection numbers on the base. This provides a “BKK -type theorem” for any choice of a cohomology class in the base $\gamma \in H^*(B, \mathbb{R})$.

We also provide an equivalent version of the generalized BKK Theorem in Section 3. It can be made as follows. Suppose the torus T has rank n and the real dimension of B is k . Similar to the toric case, a multi-polytope Δ defines a cohomology class $\rho(\Delta) \in H^2(E_{\Sigma, \Lambda}, \mathbb{R})$ on the quasitoric bundle $p: E_{\Sigma, \Lambda} \rightarrow B$. For any given j , we define a map which associates to a multi-polytope Δ a cohomology class $\gamma_{2j}(\Delta) \in H^{2j}(B, \mathbb{R})$ such that

$$\rho(\Delta)^{n+j} \cdot p^*(\gamma) = \gamma_{2j}(\Delta) \cdot \gamma,$$

for any $\gamma \in H^{k-2j}(B, \mathbb{R})$. Here “ \cdot ” denotes the cup product on the respective cohomology ring. We call the class $\gamma_{2j}(\Delta)$ the *horizontal part* of $\rho(\Delta)^{n+j}$. In Theorem 3.6, we compute the horizontal part of $\rho(\Delta)^{n+j}$ for any multi-polytope Δ .

Remark 1.1. In this paper we deal with a topological version of the BKK theorem. In particular, quasitoric bundles are locally trivial fiber bundles in smooth category over a smooth compact base. Thus, Theorems 3.5 and 3.6 are natural extensions of the topological BKK theorems for toric bundles obtained in [HKM20].

On the other hand, if one works only with algebraic toric bundles, one can use the theory of Newton-Okounkov bodies to relax assumptions on the base to be smooth or compact. We will address this direction in the future work.

Our version of the BKK theorem allows us to give a Pukhlikov-Khovanskii type description of the cohomology rings of quasitoric bundles. To do so we use the description of graded commutative algebras

which satisfy Poincaré duality obtained in [HKM20]. We then use this description to further prove a version of Stanley-Reisner and Brion’s descriptions of the cohomology ring of a quasitoric bundle in Theorems 6.4 and 6.5, respectively.

In the case when the base B is an even-dimensional manifold (e.g. a quasitoric manifold), the subring

$$A = \bigoplus H^{2i}(E_{\Sigma, \Lambda}, \mathbb{R})$$

of even-dimensional cohomology classes is a commutative Gorenstein ring. Using our version of the BKK theorem stated in Section 3 and proved in Section 4 and the exact computation of a Macaulay’s inverse system from [KM21] and recalled in Section 5, we obtain an alternative description of the ring A (see Theorem 6.3) in Section 6.

Notations

We will denote by \cdot the cup product in cohomology. For a manifold B and a cohomology class $\gamma \in H^*(B, \mathbb{R})$, we will denote by $\langle \gamma, [B] \rangle$ the pairing of γ with the fundamental class of B .

2. Generalized quasitoric manifolds

In this section we introduce generalized quasitoric manifolds and discuss various approaches to describing their cohomology.

2.1. Definition

We are going to define a notion, which is slightly more general than that of a quasitoric manifold; the methods we develop in this paper will work in this generality.

Let K be an abstract simplicial complex of dimension $n - 1$ on the vertex set $[s] = \{1, 2, \dots, s\}$. Recall that its moment-angle-complex \mathcal{Z}_K is defined to be the $(s + n)$ -dimensional cellular subspace in the unitary polydisc $(D^2)^s \subset \mathbb{C}^s$ given by the formula $\bigcup_{I \in K} \prod_{i=1}^s Y_i$, where $Y_i = D^2$, if $i \in I$ and $Y_i = S^1$, otherwise.

There is a natural (coordinatewise) action of the compact torus T^s on \mathcal{Z}_K and the orbit space \mathcal{Z}_K/T^s is homeomorphic to the cone over the barycentric subdivision of K . When P is a simple n -dimensional polytope and $K = \partial P^*$ is a polytopal sphere (that is, the boundary of the dual simplicial polytope P^*), the corresponding moment-angle-complex \mathcal{Z}_K is homeomorphic to the moment-angle manifold \mathcal{Z}_P and therefore acquires a smooth structure, since \mathcal{Z}_P is a non-degenerate intersection of Hermitian quadrics in \mathbb{C}^s due to [BPR07].

Let Σ be a complete simplicial fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ with the set of rays (1-dimensional generators) $\Sigma(1) = \{\rho_1, \dots, \rho_s\}$. The intersection K_{Σ} of the fan Σ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is, by definition, a *starshaped triangulated sphere*. Note that already in dimension 3, there exist non-polytopal starshaped spheres. Panov and Ustinovsky [PU12] proved that \mathcal{Z}_K has a smooth structure if and only if K is a starshaped sphere.

Throughout the paper we denote by $M = \text{Hom}(T^n, S^1)$ and $N = M^*$ the character and cocharacter lattices of the compact torus T^n , respectively. Suppose $K = K_{\Sigma}$ is a starshaped sphere associated with the fan Σ and $\Lambda : \Sigma(1) \rightarrow N$ is a *characteristic map*, i.e. such a map that the collection of vectors

$$\Lambda(\rho_1), \dots, \Lambda(\rho_k)$$

can be completed to a basis of the cocharacter lattice N , whenever ρ_1, \dots, ρ_k generate a cone in Σ .

Then the $(s - n)$ -dimensional subtorus $H := \ker \exp \Lambda \subset T^s$ acts freely on \mathcal{Z}_K and the smooth manifold $X_{\Sigma, \Lambda} := \mathcal{Z}_K/H$ will be called a *generalized quasitoric manifold*.

Remark 2.1. The maximal rank $s(K)$ of a freely acting subtorus in T^s is called the *Buchstaber number* of K ; it satisfies the double inequality: $1 \leq s(K) \leq s - n$. If Σ is a normal fan of a convex simple n -dimensional polytope P with s facets and $s(\partial P^*) = s - n$, then $X_{\Sigma, \Lambda}$ is a *quasitoric manifold* of P . This justifies the term ‘generalized quasitoric manifold’ we suggested above. Naturally, a quasitoric manifold serves as a key example and the most important particular case of a generalized quasitoric manifold.

Remark 2.2. The main source of examples of generalized quasitoric manifolds which are not quasitoric manifolds arises as the class of smooth, complete but non-projective toric varieties. Indeed, a smooth complete toric variety Y with an action of $(\mathbb{C}^*)^n$ can be regarded as a torus manifold with a restricted action of the compact torus $T^n = \{(z_1, \dots, z_n) \mid |z_1| = \dots = |z_n| = 1\} \subset (\mathbb{C}^*)^n$. Moreover, Y is a quasitoric manifold (i.e. the quotient space Y/T^n is diffeomorphic as a manifold with corners to a simple convex polytope) if and only if Y admits an embedding to a projective space \mathbb{P}^N . More concretely, the diffeomorphism of a quotient space Y/T^n with a simple polytope is provided by the moment map with respect to the restriction of the Fubini-Study form to Y . An explicit example of a smooth, complete, non-projective toric variety is given in [Ful93, page 71].

A closed connected codimension- $2k$ submanifold of $X_{\Sigma, \Lambda}$ is called *characteristic* if it is fixed pointwise by a subtorus of dimension k in T^n . Any such can be uniquely represented as a transverse intersection of k characteristic submanifolds of codimension 2. It is easy to see that the face poset of characteristic submanifolds in $X_{\Sigma, \Lambda}$ is isomorphic to K_Σ .

Remark 2.3. If $M = M(P, \Lambda)$ is a $2n$ -dimensional quasitoric manifold over a simple n -dimensional polytope P and $\pi: M \rightarrow P$ is a projection onto the orbit space of the smooth effective locally standard T^n -action on M , then each characteristic submanifold of codimension 2 is just the inverse image of a *facet* of P (that is, a face of codimension 1) under the map π .

In what follows we will always assume that our generalized quasitoric manifolds are *omnioriented*; as in the case of a quasitoric manifold, we say that $X_{\Sigma, \Lambda}$ is omnioriented if an orientation is specified for $X_{\Sigma, \Lambda}$ and for each of the s codimension-2 characteristic submanifolds D_i . The choice of this extra data is convenient for two reasons. First, it allows us to view the circle fixing D_i as an element in the lattice $N = \text{Hom}(S^1, T^n) \simeq \mathbb{Z}^n$. But even more importantly, the choice of omniorientation defines the fundamental class $[X_{\Sigma, \Lambda}]$ of $X_{\Sigma, \Lambda}$ as well as cohomology classes $[D_i]$ dual to the characteristic submanifolds.

We further assume that $\Sigma \subset \mathbb{R}^n$ and $N_{\mathbb{R}}$ are endowed with orientation. This defines a sign for each collection of rays $\rho_{i_1}, \dots, \rho_{i_n}$ forming a maximal cone of Σ in the following way. Let $\omega = \{i_1, \dots, i_n\}$ be a set of indices ordered in such a way that the collection of rays $\rho_{i_1}, \dots, \rho_{i_n}$ is positively oriented in \mathbb{R}^n . Then

$$\text{sign}(\omega) = \det(\Lambda(\rho_{i_1}), \dots, \Lambda(\rho_{i_n})) = \pm 1.$$

It is easy to see, that for codimension-2 characteristic submanifolds D_{i_1}, \dots, D_{i_n} in a generalized quasitoric manifold one has

$$[D_{i_1}] \cdots [D_{i_n}] = \begin{cases} \text{sign}(\omega), & \text{if } \rho_{i_1}, \dots, \rho_{i_n} \text{ form a cone in } \Sigma \\ 0, & \text{otherwise} \end{cases}$$

2.2. Cohomology ring: Stanley-Reisner-type description

In this subsection we give a description of cohomology rings of generalized quasitoric manifolds based on the notion of a Stanley-Reisner ring of a simplicial complex.

Let K be a simplicial complex on the vertex set $[s]$. The quotient algebra

$$\mathbb{Z}[K] := \mathbb{Z}[x_1, \dots, x_s]/I,$$

where I is the monomial ideal generated by all square-free monomials $x_{i_1} \cdots x_{i_t}$ with $\{i_1, \dots, i_t\} \notin K$, is called the *Stanley-Reisner ring* (or, *face ring*) of K (with integer coefficients). In what follows we assume that $\mathbb{Z}[K]$ is a graded algebra with $\deg x_i = 2, 1 \leq i \leq s$. Denote by $\mathbb{Z}[\Sigma] := \mathbb{Z}[K_\Sigma]$.

Recall that any generalized quasitoric manifold $X_{\Sigma, \Lambda}$ with $\dim \Sigma = n$ is a $2n$ -dimensional *locally standard torus manifold*; that is, a smooth connected closed orientable $2n$ -dimensional manifold with an effective smooth locally standard action of the compact torus T^n such that the fixed point set M^T is non-empty and finite.

Moreover, since the dual Q^\vee to the manifold with corners $Q := X_{\Sigma,\Lambda}/T^n$ is isomorphic to the starshaped sphere K_Σ , the orbit space Q is a *homology polytope*; that is, a manifold with corners in which all faces, including Q itself are acyclic, and all non-empty intersections of faces are connected.

The next result is a particular case of a theorem due to Masuda and Panov [MP06], who computed (equivariant) cohomology rings of torus manifolds. It describes the (equivariant) cohomology ring of $X_{\Sigma,\Lambda}$ in terms of the Stanley-Reisner ring $\mathbb{Z}[K_\Sigma]$ of K_Σ .

- Theorem 2.4.** 1. *Equivariant cohomology ring $H_{T^n}^*(X_{\Sigma,\Lambda})$ is isomorphic to $\mathbb{Z}[K_\Sigma]$;*
 2. *Ordinary cohomology ring $H^*(X_{\Sigma,\Lambda})$ is isomorphic to $\mathbb{Z}[K_\Sigma]/J$, where*

$$J = \left\langle \sum_{i=1}^s \chi(v_i)x_i : \chi \in M \right\rangle \text{ and } v_i := \Lambda(\rho_i), 1 \leq i \leq s.$$

When Σ is a normal fan of a convex simple polytope P with the Buchstaber number $s(\partial P^*) = s - n$, these results were obtained by Davis and Januszkiewicz [DJ91]. This description of the cohomology ring yields an algorithm to compute products of top-degree cohomology classes in $H^*(X_{\Sigma,\Lambda}, \mathbb{R})$. It can be done as follows.

As $R_{\Sigma,\Lambda} := H^*(X_{\Sigma,\Lambda}, \mathbb{R})$ is generated in degree 2, it suffices to consider monomials $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$. Recall that the graded piece of top degree of $R_{\Sigma,\Lambda}$ is one-dimensional and is generated by $x_{i_1} \dots x_{i_n}$, for any collection i_1, \dots, i_n of indices such that the rays $\rho_{i_1}, \dots, \rho_{i_n}$ generate a full-dimensional cone in Σ . Indeed, all such monomials $x_{i_1} \dots x_{i_n}$ yield the same element in $R_{\Sigma,\Lambda}$. Therefore, the evaluation of a monomial $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$ in the top degree amounts to expressing it as a linear combination of square free monomials.

For a monomial $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$, let $m := \sum_{i=1}^t (k_i - 1)$ be its *multiplicity*, so that being square free is equivalent to having multiplicity 0. To simplify notation, consider the monomial $x_1^{k_1} \dots x_t^{k_t}$ (always possible by reordering the variables). Suppose $x_1^{k_1} \dots x_t^{k_t}$ is a monomial with multiplicity $m > 0$ and $k_1 > 1$. The goal is to express it as a linear combination of monomials of smaller multiplicity. If ρ_1, \dots, ρ_t do not form a cone, then this monomial is equal to zero in $R_{\Sigma,\Lambda}$ and we are done. Otherwise, the set $\{v_1, \dots, v_t\}$ can be extended to a lattice basis of N , and therefore there is $\chi \in M$ such that $\chi(e_1) = 1$ and $\chi(e_j) = 0$, for $j = 2, \dots, t$. Note that χ induces a linear relation in J , so that we obtain

$$x_1^{k_1} \dots x_t^{k_t} = x_1^{k_1-1} \dots x_t^{k_t} \cdot \left(- \sum_{k=t+1}^s \chi(v_k)x_k \right).$$

This is a linear combination of monomials of multiplicity $m - 1$. Applying this procedure inductively, we end up with a linear combination of square free monomials, and thus obtain an evaluation of $x_1^{k_1} \dots x_t^{k_t}$.

2.3. Cohomology ring: volume polynomial description

Another description of the cohomology ring of $X_{\Sigma,\Lambda}$ comes from the volume polynomial on the space of analogous multi-polytopes. To a simplicial fan Σ with the characteristic map Λ one can associate a vector space of analogous multi-polytopes $\mathcal{P}_{\Sigma,\Lambda}$. Multi-fans and multi-polytopes were introduced in [HM03].

Similar to a space of virtual polytopes, the space $\mathcal{P}_{\Sigma,\Lambda}$ can be identified with the space of Λ -piecewise linear functions on Σ .

Definition 2.5. A function $f: \Sigma \rightarrow \mathbb{R}$ is called *Λ -piecewise linear* if its restriction $f|_\sigma$ to any cone σ of Σ is a pullback of a linear function on N :

$$f|_\sigma = (\Lambda^*\chi)|_\sigma, \text{ for some } \chi \in M_{\mathbb{R}}.$$

A Λ -piecewise linear function is called *integer*, if for any cone σ , the corresponding linear function $\chi_\sigma \in M$ is integer.

We define the *space of analogous multi-polytopes* as the space of all Λ -piecewise linear functions on Σ :

$$\mathcal{P}_{\Sigma, \Lambda} = \{h: \Sigma \rightarrow \mathbb{R}: \Lambda - \text{piecewise linear function on } \Sigma\}.$$

Any element of $\mathcal{P}_{\Sigma, \Lambda}$ is determined by a choice of a point on each ray from $\Sigma(1)$.

The above description generalizes the description of virtual polytopes via their support functions. To define volume and integration functionals on the space of multi-polytopes, one needs a more geometric description of multi-polytopes via convex chains studied carefully in [KLM22]. The general definition is quite technical, and is not necessary for this paper: all the properties of multi-polytopes used in this work are recalled in Section 4. However, for the better exposition we give a construction in dimension 2.

Let Σ be a two-dimensional fan with the set of rays $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ ordered such that the ray ρ_i is neighboring rays ρ_{i-1}, ρ_{i+1} . Let further $\Lambda: \Sigma(1) \rightarrow \mathbb{Z}^2$ be a characteristic map. We denote by $\lambda_1, \dots, \lambda_r$ the respective images of ρ_1, \dots, ρ_r under characteristic map. Any Λ -piecewise linear function $h \in \mathcal{P}_{\Sigma, \Lambda}$ is uniquely determined by its value on the ray generators e_1, \dots, e_r . Moreover, since Σ is a simplicial fan, every collection of values $h(e_1), \dots, h(e_r) \in \mathbb{R}$ extends to a Λ -piecewise linear function. Thus the space of Λ -piecewise linear functions is naturally isomorphic to \mathbb{R}^r .

Now we are ready to define a convex chain corresponding to a Λ -piecewise linear function h . A Λ -piecewise linear function defines a line arrangement \mathcal{A}_h in \mathbb{R}^2 via

$$L_i = \{x \in \mathbb{R}^2 \mid \langle x, \lambda_i \rangle = h(e_i)\}.$$

Note that the lines in the arrangement \mathcal{A} are ordered and cooriented by a vectors λ_i . There exists a unique broken closed broken line (s_1, \dots, s_r) with r segments such that s_i is a segment of L_i . We interpret the broken line as the image of a piecewise smooth immersion $f_h: S^1 \rightarrow \mathbb{R}^2$ of the unit circle. Then the convex chain representation of the multi-polytope defined by a Λ -piecewise linear function is given by

$$\Delta_h = \sum_U W_{f_h}(U) \cdot U,$$

where the sum is taken over the bounded regions of $\mathbb{R}^2 \setminus \bigcup L_i$ and $W_{f_h}(U)$ is a winding number of f_h with respect to any point $x \in U$. Thus we define the volume of Δ_h or an integral of a two-form ω over Δ_h as

$$\text{Vol}(\Delta_h) = \sum_U W_{f_h}(U) \text{Vol}(U) \quad \int_{\Delta_h} \omega = \sum_U W_{f_h}(U) \int_U \omega.$$

By analogy with virtual polytopes, we will usually denote a convex chain corresponding to a multi-polytope by Δ and the corresponding Λ -piecewise linear function by h_Δ (or similarly by Δ_h and h respectively).

Example 2.6. Let Σ be a normal fan to a hexagon and let $M \simeq \mathbb{Z}^2$ be a character lattice of a torus $T \simeq (S^1)^2$. We consider the characteristic pair (Σ, Λ) with $\Lambda: \Sigma(1) \rightarrow \mathbb{Z}^2$ given by

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Hence a multi-polytope $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$ is given by a six dimensional real vector of support numbers $(h_1, \dots, h_6) \in \mathbb{R}^6$. The corresponding line arrangement \mathcal{A}_h in \mathbb{R}^2 consists of six lines defined by the following six equations:

$$x_2 = h_1, \quad x_2 - x_1 = h_2, \quad x_2 = h_3, \quad x_2 - x_1 = h_4, \quad x_2 = h_5, \quad x_1 = h_6.$$

Note, that the combinatorics of \mathcal{A}_h and the corresponding multi-polytopes Δ_h depend on the concrete values of h_1, \dots, h_6 . In Figures 1a, 1b we give examples of the hyperplane arrangements \mathcal{A}_h and Δ_h for two particular choices of the values of h_1, \dots, h_6 . In particular, in the Figure 1a the chain presentation of the multi-polytope Δ_h has 2 terms, $\Delta_h = B - A$, while the chain representation of Δ_h in Figure 1b has a unique term $\Delta_h = C$.

Finally notice, that despite of the different combinatorial representation at the different regions of \mathbb{R}^6 , the volume function

$$\text{Vol}_{\Sigma, \Lambda} : \mathbb{R}^6 \rightarrow \mathbb{R}, \quad (h_1, \dots, h_6) \mapsto \text{Vol}(\Delta_h)$$

is a global polynomial in h_1, \dots, h_6 . For instance, for the combinatorial types of multi-polytopes in Figures 1a, 1b, the volume is computed as

$$\begin{aligned} \text{Vol}(\Delta_h) &= \text{Vol}(B) - \text{Vol}(A) = \frac{1}{2}(h_5 - h_3) \cdot (h_5 + h_3 - 2h_4 - 2h_6) - \frac{1}{2}(h_3 - h_1) \cdot (2h_2 + 2h_6 - h_1 - h_3) = \\ &= \frac{h_5^2}{2} - \frac{h_1^2}{2} + h_1h_2 - h_2h_3 + h_3h_4 - h_4h_5 - h_5h_6 + h_6h_1 = \text{Vol}(C). \end{aligned}$$

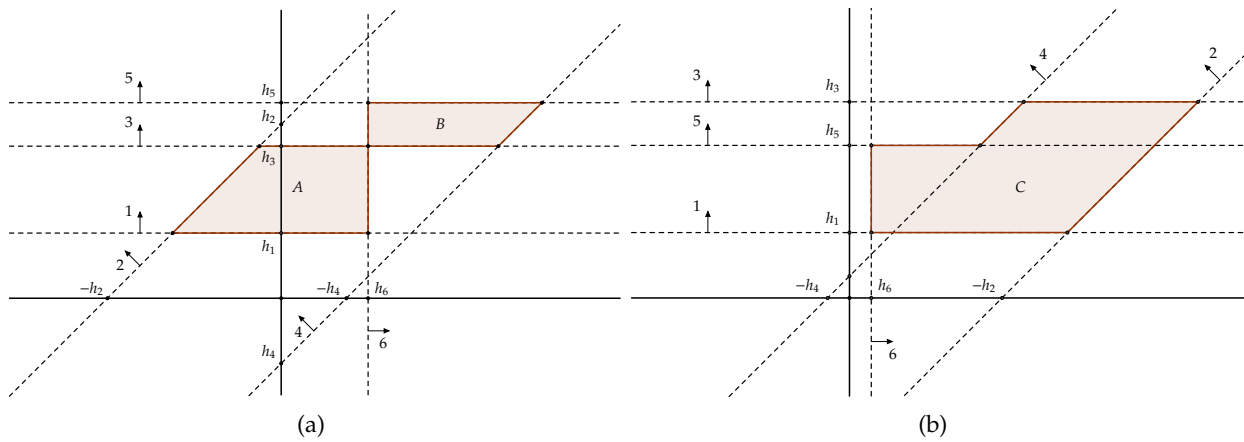


Figure 1: Two multi-polytopes corresponding to a characteristic pair (Σ, Λ) from Example 2.6.

Let $X_{\Sigma, \Lambda}$ be a generalized quasitoric manifold and let D_1, \dots, D_s be characteristic submanifolds of $X_{\Sigma, \Lambda}$ of codimension 2. Since $X_{\Sigma, \Lambda}$ is omnioriented, every multi-polytope $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$ defines a degree 2 cohomology class of $X_{\Sigma, \Lambda}$ via

$$\Delta \mapsto \sum_{i=1}^s h_{\Delta}(e_i)[D_i],$$

here $[D_i] \in H^2(X_{\Sigma, \Lambda}, \mathbb{R})$ is a class Poincaré dual to the characteristic submanifold D_i . For quasitoric manifolds, it was shown in [DJ91] that the cohomology ring $H^*(X_{\Sigma, \Lambda}, \mathbb{R})$ is generated by the classes $[D_1], \dots, [D_s]$ and, in particular, cohomology is generated by its degree-two part.

Commutative graded rings with Poincaré duality generated in degree 1 admit a convenient description via Macaulay's inverse systems. The following theorem is implicit in [PK92b] and forms a key idea to get a description of the cohomology ring $H^*(X_{\Sigma}, \mathbb{R})$. The following version is taken from [Kav11, Theorem 1.1].

Theorem 2.7. *Suppose $A = \bigoplus_{i=0}^n A_i$ is a graded finite dimensional commutative algebra over a field \mathbb{K} of characteristic 0 such that A is generated (as an algebra) by the elements A_1 of degree one, $A_0 \simeq A_n \simeq \mathbb{K}$, and the bilinear map $A_i \times A_{n-i} \rightarrow A_n$ is non-degenerate for any $i = 0, \dots, n$ (Poincaré duality). Then*

$$A \simeq \mathbb{K}[t_1, \dots, t_r] / \{p(t_1, \dots, t_r) \in \mathbb{K}[t_1, \dots, t_r] : p(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r})f(x_1, \dots, x_r) = 0\}$$

where we identify A_1 with \mathbb{K}^r via a basis v_1, \dots, v_r and define $f: A_1 \simeq \mathbb{K}^r \rightarrow \mathbb{K}$ as the polynomial given by $f(x_1, \dots, x_r) = (x_1v_1 + \dots + x_rv_r)^n \in A_n \simeq \mathbb{K}$.

Remark 2.8. Instead of identifying A_1 with \mathbb{K}^r , the previous theorem accepts a basis-free formulation in terms of $\text{Diff}(A_1)$, the ring of differential operators with constant coefficients: $A \simeq \text{Diff}(A_1)/\{p \in \text{Diff}(A_1) : p(f) = 0\}$. In particular, we formulate Theorems 5.8 and 5.9 which generalize Theorem 2.7.

Theorem 2.7 together with Theorem 2.4 imply that to compute the cohomology ring of $X_{\Sigma, \Lambda}$ it is enough to know the self-intersection index of characteristic submanifolds D_1, \dots, D_s . This was done by Ayzenberg and Masuda [AM16]. They defined the notion of a volume polynomial of a multi-polytope and obtained the following generalization of the BKK theorem to (generalized) quasitoric manifolds.

Theorem 2.9. Let $X_{\Sigma, \Lambda}$ be a (generalized) quasitoric manifold and let $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$ be a multi-polytope. Then

$$\left(\sum_{i=1}^s h_{\Delta}(e_i)[D_i] \right)^n = n! \cdot \text{Vol}(\Delta).$$

As in the case of the classical BKK Theorem, theorem 2.9 admits a polarized volume:

$$\prod_{i=1}^n \sum_{j=1}^s h_{\Delta_i}(e_j)[D_j] = n! \cdot \text{Vol}(\Delta_1, \dots, \Delta_n),$$

where $\text{Vol}(\Delta_1, \dots, \Delta_n)$ is the mixed volume.

Corollary 2.10. Let $X_{\Sigma, \Lambda}$ be a generalized quasitoric manifold, then the cohomology ring $H^*(X_{\Sigma, \Lambda}, \mathbb{R})$ can be computed as

$$H^*(X_{\Sigma, \Lambda}, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_{\Sigma, \Lambda}) / \text{Ann}(\text{Vol}),$$

where $\text{Ann}(\text{Vol})$ is the ideal of differential operators annihilating the volume polynomial on $\mathcal{P}_{\Sigma, \Lambda}$.

Example 2.11. [Continuation of Example 2.6] Consider a characteristic pair (Σ, Λ) from Example 2.6 and let $X_{\Sigma, \Lambda}$ be the corresponding quasitoric manifold. As before, the volume polynomial on the space of multi-polytopes $\mathcal{P}_{\Sigma, \Lambda} \simeq \mathbb{R}^6$ is given by

$$\text{Vol}_{\Sigma, \Lambda}(h_1, \dots, h_6) = \frac{h_5^2}{2} - \frac{h_1^2}{2} + h_1h_2 - h_2h_3 + h_3h_4 - h_4h_5 - h_5h_6 + h_6h_1.$$

It is easy to see that the ideal $\text{Ann}(\text{Vol}_{\Sigma, \Lambda})$ of differential operators annihilating $\text{Vol}_{\Sigma, \Lambda}$ is generated by 2 linear and 9 quadratic operators

$$\ell_1 = \frac{\partial}{\partial h_6} - \frac{\partial}{\partial h_2} - \frac{\partial}{\partial h_6}, \quad \ell_2 = \frac{\partial}{\partial h_1} + \dots + \frac{\partial}{\partial h_5}$$

$$q_{i,j} = \frac{\partial}{\partial h_i} \cdot \frac{\partial}{\partial h_j} \quad \text{for } |i - j| \neq 1 \pmod{6}.$$

Thus the cohomology ring of $X_{\Sigma, \Lambda}$ can be computed as

$$H^*(X_{\Sigma, \Lambda}, \mathbb{R}) = \mathbb{R} \left[\frac{\partial}{\partial h_1}, \dots, \frac{\partial}{\partial h_6} \right] / \langle \ell_1, \ell_2, q_{1,3}, \dots, q_{4,6} \rangle.$$

Note that the description above coincides with the one given by Theorem 2.4.

2.4. Cohomology ring: Brion-type description

In this subsection we give the third description of the cohomology ring of a quasitoric manifold via the ring of piecewise polynomial functions on the multi-fan (Σ, Λ) .

For a multi-fan (Σ, Λ) , we call a continuous function $f : \Sigma \rightarrow \mathbb{R}$ (Λ) -piecewise polynomial if its restriction $f|_\sigma$ to any cone σ of Σ is a pullback of a polynomial on $N_{\mathbb{R}}$ via Λ :

$$f|_\sigma = \Lambda^* g_\sigma, \quad g_\sigma \in \mathbb{R}[M].$$

The set of all piecewise polynomial functions on (Σ, Λ) forms an \mathbb{R} -algebra with respect to pointwise addition and multiplication. Let us denote the algebra of piecewise polynomial functions on (Σ, Λ) by $PP_{\Sigma, \Lambda}$. First, recall the notion of a GKM manifold.

Let $T = (\mathbb{C}^*)^n, n \geq 2$ be the algebraic n -dimensional torus. By a *GKM manifold* we mean a complex smooth projective variety M with an algebraic action of the torus T , satisfying the following conditions:

1. the fixed point set M^T is finite;
2. the set E of the complex one-dimensional orbits is finite;
3. the action is *equivariantly formal*; that is, the Leray-Serre cohomology spectral sequence for the Borel fibration

$$M \hookrightarrow ET \times_T M \rightarrow BT$$

collapses at the E_2 -term.

Suppose $\dim_{\mathbb{C}}(M) = m$. Then, due to [GKM98], the pair (M^T, E) forms an m -valent simple graph. More precisely, one has:

- The closure of each orbit $e \in E$ is an embedded $\mathbb{C}P^1$, which contains exactly two fixed points (at the north and the south poles of the Riemann sphere);
- At any $v \in M^T$, the closures of exactly m orbits meet.

Given an edge $e \in E$ connecting v and $w \in M^T$, denote by $\alpha_{vw} \in H^2(BT)$ the weight of the isotropic representation of T on $T_v \mathbb{C}P^1$, where $\mathbb{C}P^1$ is the corresponding closure of the one-dimensional T -orbit e . The graph (M^T, E) alongside with the data described above is called the *GKM graph* associated to M . The key result of [GKM98] asserts that the rational T -equivariant cohomology ring $H_T^*(M; \mathbb{Q})$ of a GKM manifold M is described in terms of the GKM graph (M^T, E) as follows.

Theorem 2.12 ([GKM98]). *One has:*

$$H_T^*(M; \mathbb{Q}) \simeq \{(f_v) \in \bigoplus_{v \in M^T} H^*(BT; \mathbb{Q}) \mid f_v - f_w \in (\alpha_{vw}), \text{ for any edge connecting } v \text{ and } w\},$$

where (α_{vw}) is the ideal generated by $\alpha_{vw} \in H^2(BT; \mathbb{Q})$.

Now we are ready to state and prove the main result of this subsection.

Theorem 2.13. *As before, let $X_{\Sigma, \Lambda}$ be a generalized quasitoric manifold given by the multi-fan (Σ, Λ) . Then equivariant cohomology ring of $X_{\Sigma, \Lambda}$ is given by the ring of piecewise polynomial functions on (Σ, Λ) :*

$$H_T^*(X_{\Sigma, \Lambda}, \mathbb{R}) \simeq PP_{\Sigma, \Lambda}.$$

Furthermore, the cohomology ring $H^*(X_{\Sigma, \Lambda}, \mathbb{R})$ is given by

$$H^*(X_{\Sigma, \Lambda}, \mathbb{R}) \simeq PP_{\Sigma, \Lambda} / \langle M \rangle.$$

Proof. A generalized quasitoric manifold $X_{\Sigma, \Lambda}$ has only finitely many T -fixed points, and for each $v \in X_{\Sigma, \Lambda}^T$ the tangent space $T_v X_{\Sigma, \Lambda}$ is decomposed as a direct sum of irreducible representations of T given by linearly independent characters. Moreover, $H^{\text{odd}}(X_{\Sigma, \Lambda}) = 0$ hence $X_{\Sigma, \Lambda}$ is an equivariantly formal GKM manifold. By the above theorem, the description of equivariant cohomology of $X_{\Sigma, \Lambda}$ follows from computation of equivariant cohomology of the corresponding GKM graph (see also [Kur09]).

Indeed, the fixed points of the T -action on $X_{\Sigma, \Lambda}$ are in bijection with maximal cones of Σ and for the two maximal cones σ_1, σ_2 connected by a codimension-1 face, the label of the corresponding edge in the GKM graph is given by a lattice generator of $(\sigma_1 \cup \sigma_2)^\perp$. Hence we have:

$$H_T^*(X_{\Sigma, \Lambda}^T, \mathbb{R}) \simeq \left\{ (f_v)_{v \in X_{\Sigma, \Lambda}^T} \in \text{Sym}(M) \times \dots \times \text{Sym}(M) \mid f_{v_{\sigma_1}}|_{\sigma_1 \cap \sigma_2} = f_{v_{\sigma_2}}|_{\sigma_1 \cap \sigma_2} \right\} \simeq \text{PP}_{\Sigma, \Lambda}$$

and

$$H^*(X_{\Sigma, \Lambda}, \mathbb{R}) \simeq \text{PP}_{\Sigma, \Lambda} / \langle M \rangle.$$

□

3. BKK theorems for quasitoric bundles

In this section, we first collect several facts on quasitoric bundles which will be used below. Then we formulate the main results of this paper: Theorems 3.5 and 3.6. Details and proofs will be given in Section 4, and in Section 6 these theorems will be used to describe cohomology rings of quasitoric bundles.

3.1. Preliminaries on quasitoric bundles

Let G be a topological group and let $p: E \rightarrow B$ be a *principal G -bundle* over a topological space B . Recall that to any principal G -bundle $p: E \rightarrow B$ and any topological space X equipped with a continuous action by G , one associates a fiber bundle by introducing a (right) action on the product $E \times X$:

$$(e, x) \cdot g := (e \cdot g, g^{-1} \cdot x).$$

The *associated fiber bundle* is given as the quotient $E \times_G X := (E \times X)/G$. It is a fiber bundle with fiber X . If $G = T \simeq (S^1)^n$ is an n -dimensional torus, then a T -principal bundle is also called a *torus bundle*.

Given a principal T^n -bundle $p: E \rightarrow B$ and a generalized quasitoric manifold $X_{\Sigma, \Lambda}$ of the real dimension $2n$, we define the corresponding *quasitoric bundle* $E_{\Sigma, \Lambda} \rightarrow B$ to be the fiber bundle associated with p ; note that its fiber is the generalized quasitoric manifold $X_{\Sigma, \Lambda}$.

Crucial to the understanding of the cohomology of fiber bundles is the following theorem (see, for instance, [Hat02, Theorem 4D.1]):

Theorem 3.1 (Leray–Hirsch). *Let E be a fiber bundle with fiber F over a compact manifold B . If there are global cohomology classes u_1, \dots, u_r on E whose restrictions $i^*(u_i)$ form a basis for the cohomology of each fiber F (where $i: F \rightarrow E$ is the inclusion), then we have an isomorphism of vector spaces:*

$$H^*(B, \mathbb{R}) \otimes H^*(F, \mathbb{R}) \rightarrow H^*(E, \mathbb{R}); \quad \sum_{i,j} b_i \otimes i^*(u_j) \mapsto \sum_{i,j} p^*(b_i) \cdot u_j.$$

Corollary 3.2. *If T is a torus, $p: E \rightarrow B$ a torus bundle as in Theorem 3.1, and X a generalized T -quasitoric manifold, then as a group the cohomology of $E_X := E \times_T X$ is given by*

$$H^*(E_X, \mathbb{R}) \simeq H^*(B, \mathbb{R}) \otimes H^*(X, \mathbb{R}).$$

Proof. As a group, the cohomology of generalized quasitoric manifolds is generated by classes Poincaré dual to characteristic submanifolds. For any characteristic submanifold X_σ of X , let $E_\sigma = E \times_T X_\sigma$ be the associated bundle which is a submanifold of E_X . Let us choose a linear basis $X_{\sigma_1}, \dots, X_{\sigma_r}$ of $H^*(X, \mathbb{R})$, then the cohomology classes u_1, \dots, u_r Poincaré dual to $E_{\sigma_1}, \dots, E_{\sigma_r}$ satisfy the condition of Theorem 3.1. The statement follows. □

Corollary 3.2 yields a description of the cohomology group of E_X . Crucial for this description is the map which associates to any characteristic submanifold X_σ of X the Poincaré dual of the corresponding T -invariant submanifold of E_X . By restricting this map to codimension-2 characteristic manifolds and extending using linearity, we obtain

$$\rho: \mathcal{P}_{\Sigma, \Lambda} \rightarrow H^2(E_X, \mathbb{R}) \quad (\text{where } (\Sigma, \Lambda) \text{ is the multi-fan of } X = X_{\Sigma, \Lambda})$$

which plays an important role in our description of the cohomology ring of E_X . We provide more details for ρ . Recall that ρ_1, \dots, ρ_r denote the rays of Σ with

$$\Lambda(\rho_1) = e_1, \dots, \Lambda(\rho_r) = e_r.$$

Let D_1, \dots, D_r be the corresponding codimension-2 characteristic submanifolds in X . We also write D_i for the submanifold $E \times_T D_i$ of E_X , note that it comes with orientation induced by omniorientation of X . Then for $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$, we have:

$$\rho(\Delta) = \sum_{i=1}^r h_\Delta(e_i)[D_i] \in H^2(E_X, \mathbb{R}),$$

where $[D_i]$ is the class Poincaré dual to $D_i \subseteq E_X$ and $h_\Delta: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is the support function of multi-polytope Δ .

As before, any character $\mu \in M$ of T can be viewed as a multi-polytope, since $\Lambda^* \mu$ is a Λ -piecewise linear function on Σ . Therefore, the map ρ defines a group homomorphism $\rho: M \rightarrow H^2(E_X, \mathbb{Z})$, which we denote by the same symbol. We extend this map by linearity to a map of vector spaces $\rho: M_{\mathbb{R}} \rightarrow H^2(E_X, \mathbb{R})$

On the other hand, any character $\lambda \in M$ defines a one-dimensional representation \mathbb{C}_λ of T , namely $t \cdot z = \lambda(t)z$ for $t \in T$, and $z \in \mathbb{C}_\lambda$. If \mathcal{L}_λ denotes the associated complex line bundle on B , i.e. $\mathcal{L}_\lambda \simeq E \times_T \mathbb{C}_\lambda$, then $\mathcal{L}_{\lambda+\mu} = \mathcal{L}_\lambda \otimes \mathcal{L}_\mu$, and thus we obtain a group homomorphism:

$$c: M \rightarrow H^2(B, \mathbb{Z}), \quad \lambda \mapsto c_1(\mathcal{L}_\lambda),$$

where $c_1(\mathcal{L}_\lambda)$ is the first Chern class. By linearity, we extend the homomorphism to a map of vector spaces:

$$c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R}).$$

The following observation about connection between $\rho(\cdot)$ and $c(\cdot)$ will be crucial for our approach.

Proposition 3.3. *Let X be a generalized quasitoric bundle given by a characteristic pair (Σ, Λ) and let $p: E_X \rightarrow B$ be as before. Then for any character $\lambda \in M$:*

$$p^*c(\lambda) = \rho(\lambda),$$

where on the right hand side of the equality λ is regarded as a virtual polytope.

Proof. A character λ of T defines an equivariant line bundle L_λ on the generalized quasitoric manifold X . The associated fiber bundle $E \times_T L_\lambda$ is a line bundle over E_X . Moreover, the Chern class of $E \times_T L_\lambda$ is given by $c_1(E \times_T L_\lambda) = \rho(\lambda)$. It is easy to see that $E \times_T L_\lambda$ is a pullback of line bundle \mathcal{L}_λ on B :

$$E \times_T L_\lambda = p^* \mathcal{L}_\lambda.$$

Hence the proposition follows from naturality of Chern classes. \square

We will finish this subsection with the following observation. The map of lattices $c: M \rightarrow H^2(B, \mathbb{Z})$ is an invariant that uniquely describes torus bundles in the topological category. Indeed, principle G -bundles over a smooth manifold B are classified by the homotopy classes of maps $f: B \rightarrow BG$ to the classifying space. Since the classifying space of a compact torus T is homotopy equivalent to $\mathbb{C}P^\infty \times \dots \times \mathbb{C}P^\infty$ and, in particular, is an Eilenberg–MacLane space $K(\mathbb{Z}^n, 2)$, homotopy classes of maps are uniquely determined by

$$c := f^*: H^2(BT) \simeq M(T) \rightarrow H^2(B)$$

We arrive at the following result.

Proposition 3.4. *Let B be a closed oriented manifold and T be a compact torus with lattice of algebraic characters $M(T)$. Then T -principal bundles over B are in bijection with homomorphisms $c: M(T) \rightarrow H^2(B, \mathbb{Z})$.*

3.2. BKK theorems for quasitoric bundles

As before, let $p: E \rightarrow B$ be a principal torus bundle with respect to a torus $T \simeq (S^1)^n$ over a compact smooth orientable manifold B of real dimension k . Let M be the character lattice of T and (Σ, Λ) be a complete multi-fan which gives rise to a generalized quasitoric manifold $X = X_{\Sigma, \Lambda}$. Let E_X be the total space of the associated quasitoric bundle. Note that E_X is a compact smooth orientable manifold of real dimension $k + 2n$. To keep notation simple we denote the projection map of the toric bundle by $p: E_X \rightarrow B$ as well.

Our main theorems show that a choice of a natural number $i \leq \frac{k}{2}$ and $\gamma \in H^{k-2i}(B, \mathbb{R})$ gives rise to a BKK -type theorem. First, we define two functions I_γ and F_γ on $\mathcal{P}_{\Sigma, \Lambda}$ as follows.

Let $f_\gamma: M_{\mathbb{R}} \rightarrow \mathbb{R}$ be given by

$$f_\gamma(x) = \langle c(x)^i \cdot \gamma, [B] \rangle,$$

where “ \cdot ” denotes the cup product of the cohomology ring $H^*(B, \mathbb{R})$. Since $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$ is a linear map, f_γ is a homogeneous polynomial of degree i on $M_{\mathbb{R}}$. This leads to the definition of I_γ :

$$I_\gamma: \mathcal{P}_{\Sigma, \Lambda} \rightarrow \mathbb{R}; \quad I_\gamma(\Delta) := \int_{\Delta} f_\gamma(x) d\mu \quad \text{for } \Delta \in \mathcal{P}_{\Sigma, \Lambda},$$

where μ denotes the Lebesgue measure on $M_{\mathbb{R}}$ normalized with respect to the lattice M , i.e. a cube spanned by an affine lattice basis of M has volume 1. By Theorem 4.1, I_γ is a well-defined homogeneous polynomial of degree $n + i$ on $\mathcal{P}_{\Sigma, \Lambda}$.

Recall the definition of $\rho: \mathcal{P}_{\Sigma, \Lambda} \rightarrow H^2(E_X, \mathbb{R})$ from Section 3.1. This leads to the definition of the function \mathcal{F}_γ :

$$\mathcal{F}_\gamma: \mathcal{P}_{\Sigma, \Lambda} \rightarrow \mathbb{R}; \quad \mathcal{F}_\gamma(\Delta) := \langle \rho(\Delta)^{n+i} \cdot p^*(\gamma), [E_X] \rangle.$$

Clearly, \mathcal{F}_γ is a homogeneous polynomial of degree $n + i$ on $\mathcal{P}_{\Sigma, \Lambda}$.

The main result of this section is the following analog of the BKK theorem for quasitoric bundles. Indeed, it expresses certain intersection numbers of cohomology classes as mixed integrals.

Theorem 3.5. *The polynomials I_γ and \mathcal{F}_γ are proportional with coefficient of proportionality given by:*

$$(n + i)! \cdot I_\gamma(\Delta) = i! \cdot \mathcal{F}_\gamma(\Delta) \quad \text{for any } \Delta \in \mathcal{P}_{\Sigma, \Lambda}.$$

In particular, the polarizations of I_γ and \mathcal{F}_γ are proportional multilinear forms, i.e. for any $\Delta_1, \dots, \Delta_{n+i} \in \mathcal{P}_{\Sigma, \Lambda}$:

$$(n + i)! \cdot I_\gamma(\Delta_1, \dots, \Delta_{n+i}) = i! \cdot \mathcal{F}_\gamma(\Delta_1, \dots, \Delta_{n+i}).$$

For the reader’s convenience, we recall the concept of polarization (or equivalently, mixed integrals). Let V be a vector space. Recall that for a homogeneous polynomial $f: V \rightarrow \mathbb{R}$ of degree m , the *polarization* of f is the unique symmetric multilinear form $g: V^m \rightarrow \mathbb{R}$ such that $g(v, \dots, v) = f(v)$. It is a well-known fact that for any vector space V and any homogeneous polynomial f of degree m , the polarization exists and can be defined as follows:

$$g(v_1, \dots, v_m) = \frac{1}{m!} L_{v_1} \dots L_{v_m} f, \tag{1}$$

where by L_v we denote the Lie derivative in direction of v . For a homogeneous polynomial $f: M_{\mathbb{R}} \rightarrow \mathbb{R}$, we will call the polarization of I_f on the space of analogous multi-polytopes $\mathcal{P}_{\Sigma, \Lambda}$ the *mixed integral* of f .

We conclude this section with an alternative interpretation of Theorem 3.5 which can be favourable for certain applications. By the Leray-Hirsch Theorem (see Corollary 3.2), $H^*(E_X, \mathbb{R}) \simeq H^*(B, \mathbb{R}) \otimes H^*(X, \mathbb{R})$, and so for $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$, the cycle $\rho(\Delta)^{n+i} \in H^{2n+2i}(E_X, \mathbb{R})$ can be written as

$$\rho(\Delta)^{n+i} = b_{2n+2i} \otimes x_0 + b_{2n+2i-2} \otimes x_2 + \dots + b_{2i+2} \otimes x_{2n-2} + b_{2i} \otimes x_{2n},$$

with $b_s \in H^s(B, \mathbb{R})$ and $x_r \in H^r(X, \mathbb{R})$. As X is a quasitoric manifold, its cohomology groups in odd degrees vanish, and therefore $x_{2k+1} = 0$ for any k . If x_{2n} is normalized such that it is dual to a point, we call b_{2i} the horizontal part of $\rho(\Delta)^{n+i}$. Equivalently, the horizontal part b_{2i} of $\rho(\Delta)^{n+i}$ is the unique class in $H^{2i}(B, \mathbb{R})$ such that

$$\langle \rho(\Delta)^{n+i} \cdot p^*(\eta), [E_X] \rangle = \langle b_{2i} \cdot \eta, [B] \rangle,$$

for any $\eta \in H^{k-2i}(B, \mathbb{R})$. Then Theorem 3.5 accepts the following reformulation.

Theorem 3.6. For any $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$, the horizontal part of $\rho(\Delta)^{n+i}$ can be computed as

$$b_{2i} = \frac{(n+i)!}{i!} \int_{\Delta} c(x)^i dx.$$

Note that $c(\cdot)^i: M_{\mathbb{R}} \rightarrow H^{2i}(B, \mathbb{R})$ is a vector valued map whose components (after choosing suitable coordinates) are given by homogeneous polynomials of degree i . Thus the integral in Theorem 3.6 exists. Furthermore, although we show that Theorem 3.5 implies Theorem 3.6, in fact they are equivalent.

Proof. Since $H^*(B, \mathbb{R})$ satisfies Poincaré duality, it suffices to check that for any $\gamma \in H^{k-2i}(B, \mathbb{R})$, we have

$$\langle i! \cdot \gamma \cdot b_{2i}, [B] \rangle = (n+i)! \left\langle \gamma \cdot \int_{\Delta} c(x)^i dx, [B] \right\rangle,$$

Recall from Theorem 3.1, that there are $u_1, \dots, u_r \in H^*(E_X, \mathbb{R})$ such that the restrictions $i^*(u_i)$ form a basis for the cohomology of each fiber X where $i: X \rightarrow E_X$ is the inclusion. In particular, there are $y_i \in \mathbb{R}u_1 \oplus \dots \oplus \mathbb{R}u_r$ such that $i^*(y_i) = x_i$. Since x_{2n} is Poincaré dual to a point (say to a torus fixed point $x \in X$), it follows that $y_{2n} = E \times_T \{x\}$, i.e. $y_{2n} = [S]^*$ is the class Poincaré dual to a section of p .

Let $[\text{pt}]^* \in H^k(B, \mathbb{R})$ be the class dual to a point in B . For $\gamma \in H^{k-2i}(B, \mathbb{R})$, let $\gamma \cdot b_{2i} = a \cdot [\text{pt}]^*$ for some real number $a \in \mathbb{R}$. Then,

$$\begin{aligned} p^*(\gamma) \cdot \rho(\Delta)^{n+i} &= p^*(\gamma) \cdot (p^*(b_{2n+2i}) \cdot y_0 + p^*(b_{2n+2i-2}) \cdot y_2 + \dots + p^*(b_{2i+2}) \cdot y_{2n-2} + p^*(b_{2i}) \cdot y_{2n}) \\ &= p^*(\gamma \cdot b_{2i}) \cdot y_{2n} = a \cdot p^*([\text{pt}]^*) \cdot E \times_T \{x\} = a \cdot [X]^* \cdot [S]^* = a \in H^{2n+k}(E_X, \mathbb{R}) \simeq \mathbb{R}, \end{aligned}$$

where $[X]^*$ is the class dual to a fiber of p . Hence, we get

$$\langle \gamma \cdot b_{2i}, [B] \rangle = \langle p^*(\gamma) \cdot \rho(\Delta)^{n+i}, [E_X] \rangle = \mathcal{F}_{\gamma}(\Delta).$$

On the other hand,

$$\left\langle \gamma \cdot \int_{\Delta} c(x)^i dx, [B] \right\rangle = \int_{\Delta} \langle \gamma \cdot c(x)^i, [B] \rangle dx = \mathcal{I}_{\gamma}(\Delta).$$

The statement follows by Theorem 3.5. \square

Like Theorem 3.5, Theorem 3.6 admits a polarized version.

4. Proof of the BKK theorems

This section is devoted to the proof of Theorem 3.5. Before we begin with the proof, let us summarize needed results on the polynomial measures on the space of analogous multi-polytopes. For the details and proofs please see [KLM22, Kho21].

As before, let (Σ, Λ) be a characteristic pair and let $\Sigma(1) = \{\rho_1, \dots, \rho_s\}$ be the set of rays of Σ as before. To a pair (Σ, Λ) we associate the space of multi-polytopes $\mathcal{P}_{\Sigma, \Lambda}$ which is identified with the space of Λ -piecewise linear functions on Σ .

Note that since the fan Σ is simplicial, a Λ -piecewise linear function on Σ is uniquely determined by its values on the unit vectors in the directions of rays of Σ . Therefore the space $\mathcal{P}_{\Sigma,\Lambda}$ is naturally isomorphic to \mathbb{R}^s with the natural coordinates h_1, \dots, h_s given by evaluation of Λ -piecewise linear functions on the unit vectors in the directions of rays ρ_1, \dots, ρ_s .

There is a correspondence between the cones of Σ and faces of the multi-polytope $\Delta \in \mathcal{P}_{\Sigma,\Lambda}$. Indeed, for each cone $\sigma \in \Sigma$ of dimension i there exists an affine subspace $M_\sigma \subset M_{\mathbb{R}}$ of codimension i with a multi-polytope Δ_σ in it. We will only use this correspondence in the case of maximal dimensional cones. Let σ be a maximal cone of Σ . Without loss of generality we may assume that σ is generated by the rays ρ_1, \dots, ρ_n . Then the corresponding affine subspace M_σ is zero-dimensional given by the system of linear equations

$$\langle \Lambda(\rho_i), x \rangle = h_\Delta(e_i), \quad i = 1, \dots, n,$$

and we identify the corresponding vertex of Δ with the unique solution of the system above.

The main result we will need is the following construction. Let $f : M_{\mathbb{R}} \rightarrow \mathbb{R}$ be a homogeneous polynomial of degree d , then there is a well-defined integration functional on the space of virtual polytopes:

$$I_f(\Delta) = \int_{\Delta} f d\mu, \quad \text{for } \Delta \in \mathcal{P}_{\Sigma,\Lambda}.^{1)}$$

The main result we will need is the following theorem.

Theorem 4.1. *Let $f : M_{\mathbb{R}} \rightarrow \mathbb{R}$ be a homogeneous polynomial of degree d , then the function $I_f : \mathcal{P}_{\Sigma,\Lambda} \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree $n + d$ on $\mathcal{P}_{\Sigma,\Lambda}$.*

We will further need the following result on the functional I_f . Let $\partial_i = \partial/\partial h_i$ be the partial derivatives along the coordinate vectors of $\mathcal{P}_{\Sigma,\Lambda} \simeq \mathbb{R}^s$.

Lemma 4.2. *Let $\omega = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset and k_1, \dots, k_r positive integers. Let $\Delta \in \mathcal{P}_{\Sigma,\Lambda}$ be a multi-polytope and $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ , then we have*

$$\partial_{i_1}^{k_1} \dots \partial_{i_r}^{k_r} (I_f|_{\mathcal{P}_{\Sigma}})(\Delta) = 0.$$

However, if $r = n$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in M_{\mathbb{R}}$, we have

$$\partial_{\omega} (I_f|_{\mathcal{P}_{\Sigma}})(\Delta) = \text{sign}(\omega) f(A) \cdot |\det(e_{i_1}, \dots, e_{i_n})|.$$

4.1. Proof of Theorem 3.5

For the reader's convenience, we recall the used notation. Let $p : E \rightarrow B$ be a principal torus bundle with respect to an n -dimensional torus T over a smooth compact orientable manifold B of real dimension k . The character lattice of T we denote by M . Let $\Sigma \subseteq \mathbb{R}^n$ be a simplicial complete fan with rays ρ_1, \dots, ρ_s and let $\Lambda : \Sigma(1) \rightarrow M$ be a characteristic map. Let $X = X_{\Sigma,\Lambda}$ be the generalized quasitoric manifold corresponding to (Σ, Λ) and denote the total space of the associated quasitoric bundle by E_X . To keep notation simple we use the same notation $p : E_X \rightarrow B$ for the projection map of the quasitoric bundle. Fix $i \leq \frac{k}{2}$ and a class $\gamma \in H^{k-2i}(B, \mathbb{R})$. The rays ρ_i correspond to codimension-2 characteristic submanifolds in X which give rise to codimension 2 submanifolds D_i in E_X . For a multi-polytope $\Delta \in \mathcal{P}_{\Sigma,\Lambda}$, we introduced

$$\rho(\Delta) = \sum_{i=1}^s h_{\Delta}(e_i) [D_i] \in H^2(E_X, \mathbb{R}),$$

where $[D_i]$ is the class dual to $D_i \subseteq E_X$ (oriented using omniorientation of $X_{\Sigma,\Lambda}$) and h_{Δ} is the support function of Δ . Further, we introduce

$$\mathcal{F}_{\gamma} : \mathcal{P}_{\Sigma,\Lambda} \rightarrow \mathbb{R}; \mathcal{F}_{\gamma}(\Delta) = \langle \rho(\Delta)^{n+i} \cdot p^*(\gamma), [E_X] \rangle.$$

We noticed that \mathcal{F}_γ is a homogeneous polynomial of degree $n + i$ on $\mathcal{P}_{\Sigma, \Lambda}$.

Recall that for any character $\lambda \in M$ we have an associated complex line bundle \mathcal{L}_λ on B . Taking Chern classes and extending by linearity, we obtain a morphism of vector spaces $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$. Let $f_\gamma: M_{\mathbb{R}} \rightarrow \mathbb{R}$ be the function $f_\gamma(x) = \langle c(x)^i \cdot \gamma, [B] \rangle$. We defined a map $\mathcal{I}_\gamma: \mathcal{P}_{\Sigma, \Lambda} \rightarrow \mathbb{R}$ to be

$$\mathcal{I}_\gamma(\Delta) = \int_{\Delta} f_\gamma(x) d\mu.$$

By Theorem 4.1, \mathcal{I}_γ is a homogeneous polynomial of degree $n + i$ on $\mathcal{P}_{\Sigma, \Lambda}$.

We prove Theorem 3.5 by induction on the parameter $0 \leq i \leq \frac{k}{2}$.

Let us start with the base case $i = 0$. If $i = 0$, then $\gamma \in H^k(B, \mathbb{R})$ is a multiple of the class dual to a point. For simplicity, let us assume that this multiple is 1. So $p^*(\gamma)$ is the class dual to a fiber, which is a generalised quasitoric manifold, and hence $\rho(\Delta)^n \cdot p^*(\gamma)$ coincides with the self-intersection index of the codimension 2 submanifold in X corresponding to Δ . By the Theorem 2.9, this can be computed by the normalized volume of Δ which equals to $n! \cdot \mathcal{I}_\gamma(\Delta)$. In other words, for $i = 0$, Theorem 3.5 reduces to Theorem 2.9.

As induction hypothesis, suppose that we know Theorem 3.5 for some $i - 1 \geq 0$. The induction step consists of proving that Theorem 3.5 is also true for i . Since both \mathcal{F}_γ and \mathcal{I}_γ are homogeneous polynomials of the same degree $n + i$, in order to show equality between $(n + i)! \cdot \mathcal{I}_\gamma(\Delta)$ and $i! \cdot \mathcal{F}_\gamma(\Delta)$, it suffices to show that all their partial derivatives of order n coincide.

In other words, it suffices to consider differential monomials $\partial_1^{k_1} \dots \partial_r^{k_r}$ where $\partial_i = \partial/\partial h_i$ are the partial derivatives along the coordinate vectors of $\mathcal{P}_{\Sigma, \Lambda} \simeq \mathbb{R}^s$ and $\sum_{i=1}^r k_i = n$. Let us call the number $\sum_{i=1}^r (k_i - 1)$ the *multiplicity of the monomial* $\partial_1^{k_1} \dots \partial_r^{k_r}$. In particular, a monomial has multiplicity 0 if and only if it is square free.

The proof of equality of the partial derivatives of order n of $(n + i)! \cdot \mathcal{I}_\gamma(\Delta)$ and $i! \cdot \mathcal{F}_\gamma(\Delta)$ is by induction on the multiplicity m of the applied differential monomial. We will refer to the induction over i as the “outer induction” and we are going to call the induction over m the “inner induction”.

The base case of the inner induction (i.e., the case of square free differential monomials) is treated in the subsections. The result of these calculations is summarized in Corollary 4.4.

4.2. Differentiation with respect to square free monomials

The results of differentiation of \mathcal{I}_γ with respect to square free differential monomials are given in Lemma 4.2. Now we verify the base case of square free differential monomials for \mathcal{F}_γ :

Lemma 4.3. *Let $\omega = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset such that $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ and k_1, \dots, k_r are positive integers. If $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$ is a multi-polytope, then we have*

$$\partial_1^{k_1} \dots \partial_r^{k_r} (\mathcal{F}_\gamma|_{\mathcal{P}_{\Sigma, \Lambda}})(\Delta) = 0.$$

If $r = n = \dim(N_{\mathbb{R}})$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in M_{\mathbb{R}}$, we have

$$\partial_\omega (\mathcal{F}_\gamma|_{\mathcal{P}_{\Sigma, \Lambda}})(\Delta) = \text{sign}(\omega) \frac{(n + i)!}{i!} f_\gamma(A).$$

Proof. Without loss of generality we may assume that $i_j = j$ for $j = 1, \dots, r$. For a monomial $\partial_1^{k_1} \dots \partial_r^{k_r}$, let $\partial_I = \partial_1 \dots \partial_r$ be the corresponding square free monomial. It is enough to show that $\partial_\omega \mathcal{F}_\gamma(\Delta) = 0$ in order to prove that $\partial_1^{k_1} \dots \partial_r^{k_r} \mathcal{F}_\gamma(\Delta) = 0$.

We compute the expansion of the polynomial \mathcal{F}_γ at Δ . This amounts to expressing $\mathcal{F}_\gamma(\Delta + \sum_{i=1}^s \lambda_i h_i)$ in terms of the monomials $\lambda_1^{\alpha_1} \dots \lambda_s^{\alpha_s}$ for non-negative integers $\alpha_1, \dots, \alpha_s$. A straightforward computation

yields:

$$\begin{aligned} \mathcal{F}_\gamma \left(\Delta + \sum_{i=1}^s \lambda_i h_i \right) &= \left(\rho(\Delta) + \sum_{i=1}^s \lambda_i [D_i] \right)^{n+i} \cdot p^*(\gamma) \\ &= \sum_{\alpha_0 + \alpha_1 + \dots + \alpha_s = n+i} \binom{n+i}{\alpha_0, \alpha_1, \dots, \alpha_s} \cdot \rho(\Delta)^{\alpha_0} \cdot [D_1]^{\alpha_1} \cdots [D_s]^{\alpha_s} \cdot p^*(\gamma) \cdot \lambda_1^{\alpha_1} \cdots \lambda_s^{\alpha_s} \\ &= \frac{(n+i)!}{(n+i-r)!} \cdot \rho(\Delta)^{n+i-r} \cdot [D_1] \cdots [D_r] \cdot p^*(\gamma) \cdot \lambda_1 \cdots \lambda_r + (\text{other terms}) \end{aligned}$$

where $\binom{n+i}{\alpha_0, \alpha_1, \dots, \alpha_s} = \frac{(n+i)!}{\alpha_0! \alpha_1! \cdots \alpha_s!}$ denotes the usual multinomial coefficient. The derivative $\partial_\omega \mathcal{F}_\gamma(\Delta)$ is equal to the coefficient in front of the monomial $\lambda_1 \cdots \lambda_r$ in the expression $\mathcal{F}_\gamma(\Delta + \sum_{i=1}^s \lambda_i h_i)$:

$$\partial_\omega \mathcal{F}_\gamma(\Delta) = \frac{(n+i)!}{(n+i-r)!} \left\langle \rho(\Delta)^{n+i-r} \cdot [D_1] \cdots [D_r] \cdot p^*(\gamma), [E_X] \right\rangle.$$

Characteristic submanifolds D_1, \dots, D_r intersect transversely in E_X , so the product $[D_1] \cdots [D_r]$ is the class Poincaré dual to their intersection. In the case that e_1, \dots, e_r do not generate a cone in Σ the such intersection is empty, and so $\partial_\omega \mathcal{F}_\gamma(\Delta) = 0$.

For the proof of the second part, let $r = n = \dim(N_{\mathbb{R}})$. We have

$$\partial_\omega \mathcal{F}_\gamma(\Delta) = \frac{(n+i)!}{i!} \left\langle \rho(\Delta)^i \cdot [D_1] \cdots [D_n] \cdot p^*(\gamma), [E_X] \right\rangle.$$

If e_1, \dots, e_n generate a cone in Σ dual to the vertex A of Δ , then $[D_1] \cdots [D_n] = [E_A]$, where $E_A = E \times_T A$ is the torus invariant submanifold in E_X corresponding to A . Note that the omniorientation of X defines an orientation of E_A , hence the class $[E_A]$ is well-defined. In particular, the restriction of the projection map $p: E_A \rightarrow B$ is a diffeomorphism which is orientation preserving if $\text{sign}(\omega) = 1$ and orientation reversing if $\text{sign}(\omega) = -1$.

Now, let $\tilde{\Delta} = \Delta - A$ be a multi-polytope which is the translation of the multi-polytope Δ for which the vertex A is at the origin. Since the vertex of $\tilde{\Delta}$ corresponding to A is at the origin, we get

$$h_{\tilde{\Delta}}(e_1) = \dots = h_{\tilde{\Delta}}(e_n) = 0, \quad \text{and so} \quad \rho(\tilde{\Delta}) = \sum_{j>n} h_{\tilde{\Delta}}(e_j) \cdot [D_j].$$

Hence, by the first part, we get $\rho(\tilde{\Delta}) \cdot [D_1] \cdots [D_n] = 0$ as there is no cone in Σ with more than n rays. Therefore:

$$\rho(\Delta)^i \cdot [D_1] \cdots [D_n] \cdot p^*(\gamma) = \rho(\tilde{\Delta} + A)^i \cdot [D_1] \cdots [D_n] \cdot p^*(\gamma) = \rho(A)^i \cdot [E_A] \cdot p^*(\gamma).$$

By Proposition 3.3, $\rho(A) = p^*c(A)$. Since $p: E_A \rightarrow B$ is a diffeomorphism, we get:

$$\left\langle \rho(A)^i \cdot p^*(\gamma) \cdot [E_A], [E_X] \right\rangle = \left\langle (p^*c(A))^i \cdot p^*(\gamma) \cdot [E_A], [E_X] \right\rangle = \left\langle c(A)^i \cdot \gamma, \text{sign}(\omega)[B] \right\rangle = \text{sign}(\omega) f_\gamma(A),$$

and therefore $\partial_\omega \mathcal{F}_\gamma(\Delta) = \text{sign}(\omega) \frac{(n+i)!}{i!} f_\gamma(A)$. \square

The base case of the inner induction is an immediate corollary of Lemmas 4.2 and 4.3:

Corollary 4.4 (Base case of the inner induction). *For any $i \leq \frac{k}{2}$, any $\gamma \in H^{k-2i}(B, \mathbb{R})$ and any square free differential monomial $\partial_\omega = \partial_{i_1} \cdots \partial_{i_n}$ of order n (where $\omega = \{i_1, \dots, i_n\} \subseteq \{1, \dots, s\}$), we have:*

$$\partial_\omega \left((n+i)! \cdot \mathcal{I}_\gamma(\Delta) \right) = \partial_\omega \left(i! \cdot \mathcal{F}_\gamma(\Delta) \right).$$

4.3. The inner induction step

In this subsection, the index set $\omega \subseteq \{1, \dots, s\}$ is considered as being a multiset. As the induction hypothesis, suppose that $\partial_\omega((n+i)! \cdot \mathcal{I}_\gamma(\Delta)) = \partial_\omega(i! \cdot \mathcal{F}_\gamma(\Delta))$, for each differential monomial ∂_i of multiplicity $m-1 \geq 0$. It remains to show that the equality is true for differential monomials of multiplicity m .

As before, to keep notation simple assume $i_j = j$ for $j = 1, \dots, r$, so that $\partial_\omega = \partial_1^{k_1} \dots \partial_r^{k_r}$ for positive integers k_1, \dots, k_r . By reordering the coordinates of $\mathcal{P}_{\Sigma, \Lambda}$, we may assume $k_1 > 1$. By Lemmas 4.2 and 4.3, it suffices to consider the case where the vectors e_1, \dots, e_r form a cone in Σ (as otherwise $\partial_\omega \mathcal{I}_\gamma(\Delta) = 0 = \partial_\omega \mathcal{F}_\gamma(\Delta)$).

We are going to express ∂_1 as a combination of Lie derivative L_v (for some $v \in M_{\mathbb{R}}$) and other partial derivatives. Then the (inner) induction step will follow by an explicit computation of $L_v \mathcal{I}_\gamma(\Delta)$ and $L_v \mathcal{F}_\gamma(\Delta)$.

As e_1, \dots, e_r form a cone in the smooth fan Σ , they can be completed to a basis of $N_{\mathbb{R}}$. We take v to be the first vector of the dual basis of $M_{\mathbb{R}}$. Then $\langle v, e_1 \rangle = 1$, and $\langle v, e_j \rangle = 0$ for $j = 2, \dots, r$. Since $v \in M_{\mathbb{R}}$ (considered as an element of $\mathcal{P}_{\Sigma, \Lambda}$) is given by $v = \sum_{i=1}^s \langle v, e_i \rangle h_i$, we obtain that $L_v = \sum_{i=1}^s \langle v, e_i \rangle \partial_i$, and thus $\partial_1 = L_v - \sum_{j>r} \langle v, e_j \rangle \partial_j$. We get:

$$\begin{aligned} \partial_\omega &= \partial_1^{k_1} \dots \partial_r^{k_r} = \left(L_v - \sum_{j>r} \langle v, e_j \rangle \partial_j \right) \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \\ &= L_v \cdot \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} - \sum_{j>r} \langle v, e_j \rangle \cdot \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j. \end{aligned}$$

Since $k_1 > 1$ and $j > r$, each monomial in the sum $\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j$ has multiplicity $m-1$, so by the inner induction hypothesis, we get

$$(n+i)! \cdot \left(\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j \right) \cdot \mathcal{I}_\gamma(\Delta) = i! \cdot \left(\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j \right) \cdot \mathcal{F}_\gamma(\Delta).$$

It remains to consider the first summand:

Lemma 4.5. *In the situation above, we have $(n+i)! \cdot L_v \mathcal{I}_\gamma(\Delta) = i! \cdot L_v \mathcal{F}_\gamma(\Delta)$.*

Proof. A direct calculation shows:

$$\begin{aligned} L_v \mathcal{I}_\gamma(\Delta) &= \partial_t \Big|_{t=0} \left(\int_{\Delta+tv} c(x)^i \cdot \gamma \, d\mu \right) = \partial_t \Big|_{t=0} \left(\int_{\Delta} c(x+tv)^i \cdot \gamma \, d\mu \right) = \\ &= \partial_t \Big|_{t=0} \left(\int_{\Delta} \left(\sum_{a=0}^i \binom{i}{a} t^a \cdot c(v)^a c(x)^{i-a} \right) \cdot \gamma \, d\mu \right) = \int_{\Delta} i \cdot c(v) c(x)^{i-1} \cdot \gamma \, d\mu \\ &= i \cdot \int_{\Delta} c(x)^{i-1} \cdot (c(v) \cdot \gamma) \, d\mu = i \cdot \mathcal{I}_{c(v), \gamma}(\Delta). \end{aligned}$$

Similarly, by a direct calculation and Proposition 3.3, we get

$$\begin{aligned} L_v \mathcal{F}_\gamma(\Delta) &= \partial_t \Big|_{t=0} \left(\rho(\Delta + tv)^{n+i} \cdot p^*(\gamma) \right) = \partial_t \Big|_{t=0} \left((\rho(\Delta) + t\rho(v))^{n+i} \cdot p^*(\gamma) \right) = \\ &= \partial_t \Big|_{t=0} \left(\left(\sum_{a=0}^{n+i} \binom{n+i}{a} t^a \cdot \rho(v)^a \cdot \rho(\Delta)^{n+i-a} \right) \cdot p^*(\gamma) \right) = (n+i) \cdot \rho(v) \cdot \rho(\Delta)^{n+i-1} \cdot p^*(\gamma) \\ &= (n+i) \cdot \rho(\Delta)^{n+i-1} \cdot p^*(c(v) \cdot \gamma) = (n+i) \cdot \mathcal{F}_{c(v), \gamma}(\Delta). \end{aligned}$$

Since $c(v) \cdot \gamma \in H^{k-2(i-1)}(B, \mathbb{R})$, by the induction hypothesis of the outer induction, we have

$$(n+i-1)! \cdot \mathcal{I}_{c(v), \gamma}(\Delta) = (i-1)! \cdot \mathcal{F}_{c(v), \gamma}(\Delta).$$

Therefore,

$$(n + i)! \cdot L_v \mathcal{I}_\gamma(\Delta) = (n + i)! \cdot i \cdot \mathcal{I}_{c(v),\gamma}(\Delta) = i! \cdot (n + i) \cdot \mathcal{F}_{c(v),\gamma}(\Delta) = i! \cdot L_v \mathcal{F}_\gamma(\Delta).$$

□

5. Graded-commutative n-self-dual algebras

In this section we briefly summarize some algebraic results obtained in the recent papers [HKM20] and [KM21] that we are going to rely on when describing cohomology rings of quasitoric bundles in a way similar to the classical approach by Khovanskii and Pukhlikov [PK92a].

5.1. The general case

Here, we introduce all the necessary algebraic preliminaries that will be used to describe the entire cohomology rings of quasitoric bundles. Our presentation follows that in [HKM20].

Recall that a graded \mathbb{K} -algebra $A = A^0 \oplus A^1 \oplus \dots \oplus A^k \oplus \dots$ over a field \mathbb{K} is called *graded-commutative* if for any pair of homogeneous elements x, y , the following relation holds:

$$xy = (-1)^{\deg(x)\deg(y)}yx$$

where $\deg(x)$ (resp. $\deg(y)$) denotes the degree of x (resp. of y).

Definition 5.1. Let n be a natural number. A graded-commutative \mathbb{K} -algebra A is said to be *Poincaré n-self-dual* (or just, *n-self-dual*) if the following conditions hold:

1. The algebra A has a multiplicative unit element e which is homogeneous of degree zero, i.e. $e \in A^0 \subseteq A$.
2. The homogeneous components A^k for $k > n$ vanish, i.e. $A^k = 0$ for $k > n$, and $\dim_{\mathbb{K}}(A^n) = 1$.
3. The pairing $A^k \times A^{n-k} \rightarrow A^n$ induced by multiplication in the algebra A is non-degenerate for $0 \leq k \leq n$.

Here is a key example we are most interested in throughout this paper.

Example 5.2. Let M be a connected compact oriented n -dimensional manifold. By Poincaré duality, the cohomology ring $A = H^*(M, \mathbb{K})$ is an n -self-dual graded-commutative \mathbb{K} -algebra. Moreover, A is equipped with two extra structures:

1. A \mathbb{K} -linear function $\ell_* : A \rightarrow \mathbb{K}$ given as follows. For $\alpha \in A^n$, we let $\ell_*(\alpha)$ be equal to the value of the cohomology class α on the fundamental class of the oriented manifold M . For $\alpha \in A^k, k \neq n$, we set $\ell_*(\alpha) = 0$. We extend ℓ_* to the whole A by linearity.
2. A \mathbb{K} -bilinear intersection form F_{ℓ_*} on A defined by the identity $F_{\ell_*}(\alpha, \beta) = \ell_*(\alpha \cdot \beta)$ (recall that we denote the cup product in the cohomology ring A by “ \cdot ”).

Remark 5.3. Note that F_{ℓ_*} is non-degenerate; in fact, the non-degeneracy of F_{ℓ_*} is equivalent to the Poincaré duality on A .

Let $C = C^0 \oplus C^1 \oplus \dots \oplus C^k \oplus \dots$ be a graded-commutative \mathbb{K} -algebra with multiplicative unit element $e \in C^0$ and $\dim_{\mathbb{K}}(C^0) = 1$. Our goal is to describe all n -self dual factor-algebras of C . To that extent we introduce the following notion.

Definition 5.4. An ideal $\mathfrak{I} \subseteq C$ is called *n-self-dual* (or just, *n-sd ideal*) if \mathfrak{I} is a two-sided homogeneous ideal in C and the factor-algebra $A = C/\mathfrak{I}$ is n -self-dual.

It is easy to see that if $\mathfrak{I} \subseteq C$ is an n -sd ideal, then $L = \mathfrak{I} \cap C^n$ is a hyperplane in C^n . As hyperplanes are in correspondence with linear functions, we introduce the next definition.

Definition 5.5. A linear function $\ell : C \rightarrow \mathbb{K}$ on a graded-commutative \mathbb{K} -algebra C is *n-homogeneous* if it is not identically equal to zero on C , but for any $m \neq n$ its restriction to the homogeneous component C^m vanishes.

Denote by $L_\ell \subseteq C^n$ the hyperplane in C^n defined by the identity

$$L_\ell := \{\ell = 0\} \cap C^n.$$

Observe that any hyperplane $L \subseteq C^n$ is equal to a hyperplane of the type L_ℓ for a unique (up to a non-zero scalar multiple) n -homogeneous linear function ℓ .

In the rest of this subsection, we explain how to obtain an n -self-dual factor-algebra from an n -homogeneous linear function $\ell: C \rightarrow \mathbb{K}$.

Definition 5.6. For an n -homogeneous linear function $\ell: C \rightarrow \mathbb{K}$ let $\mathfrak{I}_1(L_\ell)$ and $\mathfrak{I}_2(L_\ell)$ be the subsets of C defined by the following conditions:

1. an element $a \in C$ belongs to $\mathfrak{I}_1(L_\ell)$ if and only if $\ell(ab) = 0$ for all $b \in C$;
2. an element $b \in C$ belongs to $\mathfrak{I}_2(L_\ell)$ if and only if $\ell(ab) = 0$ for all $a \in C$.

For any hyperplane $L_\ell \subseteq C^n$ one has:

$$\mathfrak{I}_1(L_\ell) = \mathfrak{I}_2(L_\ell) =: \mathfrak{I}(L_\ell) \text{ is an } n\text{-sd ideal in } C,$$

where ℓ is an n -homogeneous linear function. Note that the ideals $\mathfrak{I}_1(L_\ell)$ and $\mathfrak{I}_2(L_\ell)$ depend only on the hyperplane $L := L_\ell$, and so the notation $\mathfrak{I}(L) := \mathfrak{I}(L_\ell)$ makes sense.

Definition 5.7. Let $\ell: C \rightarrow \mathbb{K}$ be an n -homogeneous linear function on a graded-commutative algebra C . The Frobenius bilinear form $F_\ell: C \times C \rightarrow \mathbb{K}$ associated with ℓ is the form defined by the identity $F_\ell(a, b) = \ell(a \cdot b)$.

It is easy to see that the n -sd ideal $\mathfrak{I}(L)$ coincides with the left and also the right kernel of F_ℓ for $L = L_\ell$.

It turns out that the set of n -sd factor-algebras $A = C/\mathfrak{I}$ equipped with non-degenerate intersection forms is in one-to-one correspondence with the set of n -homogeneous linear functions on C . More precisely, the following fundamental result takes place.

Theorem 5.8. Let C be a graded-commutative algebra over a field \mathbb{K} , and let A be an n -sd factor-algebra of C with a chosen isomorphism $\phi: A^n \rightarrow \mathbb{K}$. Let $\rho: C \rightarrow A$ be the natural epimorphism of algebras. Then

$$A \simeq C/\mathfrak{I}(L_\ell),$$

where ℓ is the n -homogeneous linear function on C defined by

$$\ell: C^n \rightarrow \mathbb{K}, \quad \ell(b) = \phi(\rho(b)),$$

and extended by 0 to C^i with $i \neq n$.

Moreover, for any n -homogeneous linear function ℓ , the algebra $A = C/\mathfrak{I}(L_\ell)$ is n -self dual and the Frobenius form F_ℓ induces a non-degenerate pairing on A .

5.2. The even-dimensional case

In this subsection, we are going to introduce the algebraic results we need in order to obtain a more convenient way to describe the subring $A^*(E_X)$ of all even-dimensional classes in the cohomology ring $H^*(E_X)$ of a quasitoric bundle E_X with an even-dimensional base B . Our presentation follows that in [KM21].

Let \mathbb{K} be a field of zero characteristic, in this paper we assume $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, or \mathbb{C} . Throughout this subsection A will be a $2n$ -self-dual graded-commutative algebra over \mathbb{K} with vanishing odd-degree part (therefore, A is a commutative algebra in the usual sense).

Let $W \subset A$ be a vector subspace, which multiplicatively generates A . Then W has a natural graded vector space structure: $W = \bigoplus_{k=0}^n W_{2k}$, where $W_{2k} := W \cap A^{2k}$.

Let $\pi: V \rightarrow \bigoplus_{k=1}^n W_{2k}$ be a surjective linear mapping for a finite-dimensional vector space V . This makes V into a graded vector space via $V = \bigoplus_{k=1}^n V_{2k}$, where $V_{2k} := \pi^{-1}(W_{2k})$.

Observe that π extends to an epimorphism of algebras

$$\pi: \text{Sym}(V) \rightarrow A.$$

Furthermore, the grading on V gives rise to a grading on $\text{Sym}(V)$:

$$\deg(v_1^{i_1} \dots v_n^{i_n}) = 2 \sum_{k=1}^n ki_k, \text{ for any } v_k \in V_{2k},$$

making π into a graded morphism.

Firstly, we are going to describe the ideal $\mathfrak{S}_\ell \subset \text{Sym}(V)$ such that

$$A \simeq \text{Sym}(V)/\mathfrak{S}_\ell.$$

Here, ℓ is defined on $\text{Sym}(V)$ as a composition of maps:

$$\ell = \alpha \circ \pi: \text{Sym}(V) \rightarrow \mathbb{K},$$

where $\alpha: A \rightarrow \mathbb{K}$ is a $2n$ -homogeneous linear function such that $\alpha: A^{2n} \simeq \mathbb{K}$. Note that ℓ is a $2n$ -quasi-homogeneous linear function.

Let us denote by \mathfrak{S}_ℓ the following ideal in $\text{Sym}(V)$:

$$\mathfrak{S}_\ell = \{P \in \text{Sym}(V) \mid \ell(P \cdot Q) = 0 \text{ for any } Q \in \text{Sym}(V)\}.$$

By Theorem 5.8 (see also [KKE21] for similar results), the kernel of the graded epimorphism of algebras $\pi: \text{Sym}(V) \rightarrow A$ is given by the ideal just defined: $\ker(\pi) = \mathfrak{S}_\ell$, and hence

$$A \simeq \text{Sym}(V)/\mathfrak{S}_\ell.$$

Below we describe the ideal \mathfrak{S}_ℓ in an alternative way, which is more useful in some applications.

Observe that the symmetric algebra $\text{Sym}(V)$ can be identified with the space $\text{Diff}(V)$ of differential operators with constant coefficients on V as follows. The zero degree component $\text{Sym}_0(V) \simeq \mathbb{K}$ corresponds to operators of multiplication by a number. For positive degree elements one can define a map

$$\mathcal{D}: v_1 \dots v_k \mapsto L_{v_1} \dots L_{v_k},$$

where L_v is the Gateaux derivative in the direction $v \in V$. For an element $P \in \text{Sym}(V)$ we denote by $\mathcal{D}_P \in \text{Diff}(V)$ the corresponding differential operator.

Define a map

$$\text{Exp}^*: \text{Sym}(V)^\vee \rightarrow \mathbb{K}[[V]], \quad \text{Exp}^*: \alpha \mapsto \sum_{i=0}^{\infty} \alpha \left(\frac{x^i}{i!} \right).$$

It turns out that $\text{Exp}^*(\ell)$ is a quasi-homogeneous polynomial of degree $2n$ for the $2n$ -quasi-homogeneous linear function ℓ .

Now we are ready to give an alternative description for the ideal \mathfrak{S}_ℓ in $\text{Sym}(V)$.

Theorem 5.9. *Let A be a $2n$ -self-dual graded-commutative algebra over \mathbb{K} with vanishing odd-degree part. Let V be a finite-dimensional graded vector space over \mathbb{K} and $\pi: \text{Sym}(V) \rightarrow A$ be a graded epimorphism of algebras.*

Suppose $\ell: \text{Sym}(V) \rightarrow \mathbb{K}$ is a $2n$ -quasi-homogeneous linear function. Then we have

$$A \simeq \text{Sym}(V)/\text{Ann}(\text{Exp}^*(\ell)),$$

where

$$\text{Ann}(\text{Exp}^*(\ell)) := \{P \in \text{Sym}(V) \mid \mathcal{D}_P(\text{Exp}^*(\ell)) = 0\}.$$

We call the quasi-homogeneous polynomial $\text{Exp}^*(\ell)$ the *potential* of A and denote it by P_A .

6. Applications

6.1. Pukhlikov-Khovanskii description of cohomology ring of a quasitoric bundle

In this section, we apply the results from Sections 4 and 5 to give a Pukhlikov-Khovanskii-type description of the cohomology ring of a quasitoric bundle that generalizes the descriptions obtained in [AM16] for quasitoric manifolds and in [HKM20] for toric bundles.

Let, as before, $p: E \rightarrow B$ be a principal T -bundle, Σ a complete simplicial fan with $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$, $X = X_{\Sigma, \Lambda}$ a generalized quasitoric manifold, and E_X the corresponding quasitoric bundle.

By the Leray-Hirsch theorem (see Theorem 3.1) the cohomology ring $H^*(E_X, \mathbb{R})$ is a quotient of the polynomial algebra $S[x_1, \dots, x_r]$, where $S = H^*(B, \mathbb{R})$ is the cohomology ring of the base.

Theorem 6.1. *In the notation from above, the cohomology ring of E_X is given by*

$$H^*(E_X, \mathbb{R}) \simeq S[x_1, \dots, x_r] / \mathfrak{I}(L_\ell)$$

where $\ell: S[x_1, \dots, x_r] \rightarrow \mathbb{R}$ is a $(k + 2n)$ -homogeneous linear function defined by:

$$\ell(\gamma \cdot x_{i_1} \cdots x_{i_s}) = \mathcal{I}_\gamma(\rho_{i_1}, \dots, \rho_{i_s})$$

for any monomial $\gamma \cdot x_{i_1} \cdots x_{i_s}$ with $\deg(\gamma) + 2s = k + 2n$.

Proof. By Theorem 5.8, as $H^*(E_X, \mathbb{R})$ is a graded commutative $(k + 2n)$ -self dual factor algebra of $S[x_1, \dots, x_r]$, it is given by $S[x_1, \dots, x_r] / \mathcal{I}(L_\ell)$ for a $(k + 2n)$ -homogeneous linear function ℓ which is obtained by pairing cohomology classes with the fundamental class of E_X . Hence the statement follows by Theorem 3.5. \square

For the next two theorems, let us assume that $\dim(B) = 2k$ is even. In this case, $\dim E_X = 2(k + n)$ is also even. Then let us denote by $A^*(B), A^*(E_X)$ the corresponding rings of even cohomology classes:

$$A^*(B) = \bigoplus_{i=0}^{2k} H^{2i}(B, \mathbb{R}), \quad A^*(E_X) = \bigoplus_{i=0}^{2(k+n)} H^{2i}(E_X, \mathbb{R}).$$

Rings $A^*(B), A^*(E_X)$ are commutative graded rings satisfying Poincaré duality. Hence we can use Theorem 5.9 to obtain their presentations.

Denote by P_B the potential of $A^*(B)$ with respect to the generating vector space $A^*(B)$ and by P_{E_X} the potential of $A^*(E_X)$ with respect to $A^*(B) \oplus \mathcal{P}_{\Sigma, \Lambda}$. The following theorem provides a relation between potentials P_{E_X} and P_B .

Theorem 6.2. *For any $\Delta \in \mathcal{P}_{\Sigma, \Lambda}$ and $\gamma \in A^*(B)$ we have*

$$P_{E_X}(\gamma, \Delta) = \int_{\Delta} P_B(c(\lambda) + \gamma) d\lambda.$$

Proof. The proof relies on Theorem 3.6 and will be obtain by a direct calculation. Recall that by Theorem 5.9, the potentials P_{E_X}, P_B are given by

$$P_{E_X} = \text{Exp}^*\langle [E_X], \cdot \rangle, \quad P_B = \text{Exp}^*\langle [B], \cdot \rangle$$

In other words, the evaluations of potentials P_{E_X} and P_B on elements of $A^*(B) \oplus \mathcal{P}_{\Sigma, \Lambda}$ and $A^*(B)$ respectively are given by the following formulas

$$P_{E_X}(\gamma, \Delta) = \left\langle [E_X], \sum_{j=0}^{\infty} \frac{(p^*(\gamma) + \rho(\Delta))^j}{j!} \right\rangle, \quad P_B(\gamma) = \left\langle [B], \sum_{j=0}^{\infty} \frac{\gamma^j}{j!} \right\rangle$$

By definition of horizontal parts $h_i(\Delta)$ of Δ we have

$$P_{E_X}(\gamma, \Delta) = \left\langle [E_X], p^*(\exp(\gamma)) \cdot \sum_{j=0}^{\infty} \frac{\rho(\Delta)^j}{j!} \right\rangle = \left\langle [B], \exp(\gamma) \cdot \sum_{j=0}^{\infty} \frac{h_j(\Delta)}{j!} \right\rangle.$$

Now, by Theorem 3.6 we get $h_j(\Delta) = 0$ for $j < n$ and

$$h_{n+i}(\Delta) = \frac{(n+i)!}{i!} \int_{\Delta} c(\lambda)^i d\lambda,$$

otherwise. Hence we obtain

$$\begin{aligned} P_{E_X}(\gamma, \Delta) &= \left\langle [B], \exp(\gamma) \cdot \sum_{j=0}^{\infty} \frac{h_j(\Delta)}{j!} \right\rangle = \left\langle [B], \exp(\gamma) \cdot \sum_{i=0}^{\infty} \frac{h_{n+i}(\Delta)}{(n+i)!} \right\rangle = \\ &= \left\langle [B], \exp(\gamma) \cdot \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\Delta} c(\lambda)^i d\lambda \right\rangle = \left\langle [B], \int_{\Delta} \exp(\gamma + c(\lambda)) d\lambda \right\rangle = \\ &= \int_{\Delta} \langle [B], \exp(c(\lambda) + \gamma) \rangle d\lambda = \int_{\Delta} P_B(c(\lambda) + \gamma) d\lambda. \end{aligned}$$

This finishes the proof. \square

The next result is a direct generalization of [KM21, Theorem 5.8] to quasitoric bundles.

Theorem 6.3. *In the above notation, suppose $\dim B$ is even. Then the ring of even-degree cohomology classes of a quasitoric bundle is given by*

$$A^*(E_X) = \text{Sym}(A^*(B) \oplus \mathcal{P}_{\Sigma, \Lambda}) / \text{Ann}(P_{E_X}),$$

where the quasi-homogeneous polynomial P_{E_X} is given by

$$P_{E_X}(\gamma, \Delta) = \int_{\Delta} P_B(c(\lambda) + \gamma) d\lambda.$$

Proof. The proof goes by a direct application of Theorem 5.9 and Theorem 6.2. \square

6.2. Stanley-Reisner description of cohomology ring of a quasitoric bundle

In this subsection we provide a generalization of the Stanley-Reisner description for the cohomology ring to quasitoric bundle case. A version of Theorem 6.4 which works for more general torus manifold bundles was obtained in [DKU19].

Theorem 6.4. *Let E_X be a quasitoric bundle associated with a generalized quasitoric manifold $X = X_{\Sigma, \Lambda}$ as before. Then $H^*(E_X, \mathbb{R})$ is isomorphic (as an $H^*(B, \mathbb{R})$ -algebra) to the quotient of $H^*(B, \mathbb{R})[x_1, \dots, x_r]$ by*

$$\left(\left\langle x_{j_1} \cdots x_{j_k} : \rho_{j_1}, \dots, \rho_{j_k} \text{ do not span a cone of } \Sigma \right\rangle + \left\langle c(\lambda) - \sum_{i=1}^n \langle e_i, \lambda \rangle x_i : \lambda \in M \right\rangle \right).$$

Proof. The proof is identical to the proof of [HKM20, Theorem 3.5]. \square

Note the similarities with the Stanley-Reisner description of the cohomology ring of toric varieties and quasitoric manifolds. Indeed, the first ideal in Theorem 6.4 corresponds to the Stanley-Reisner ideal of the corresponding toric variety.

6.3. Brion description of cohomology ring of a quasitoric bundle

In this subsection we give a proof of a generalization of the Brion description for a cohomology ring to the quasitoric bundle case.

Theorem 6.5. *Let E_X be a quasitoric bundle associated with a generalized quasitoric manifold $X = X_{\Sigma, \Lambda}$ as before. Then $H^*(E_X, \mathbb{R})$ is isomorphic (as an $H^*(B, \mathbb{R})$ -algebra) to the quotient of $H^*(B, \mathbb{R}) \otimes \text{PP}_{\Sigma, \Lambda}^1$ by the ideal*

$$\langle c(\lambda) \otimes 1 - 1 \otimes \langle \cdot, \lambda \rangle : \lambda \in M \rangle,$$

where $\langle \cdot, \lambda \rangle \in \text{PP}_{\Sigma, \Lambda}^1$ is a (Λ -piecewise) linear function on Σ .

Proof. For the generalized quasitoric manifold X , Theorem 2.4 (1) and Theorem 2.13 imply that

$$\mathbb{R}[\Sigma] \simeq H_T^*(X; \mathbb{R}) \simeq \text{PP}_{\Sigma, \Lambda}.$$

It is easy to see that the composition isomorphism of \mathbb{R} -algebras is given by the mapping $v_i \mapsto \phi_i$, for all $1 \leq i \leq s$. Hence the set of piecewise linear functions $\{\phi_i : 1 \leq i \leq s\}$ forms a basis of $\text{PP}_{\Sigma, \Lambda}^1$.

Then any Λ -piecewise linear function $\langle \cdot, \lambda \rangle$ on Σ can be uniquely expressed as a linear combination of elements of this basis, namely

$$\langle \cdot, \lambda \rangle = \sum_{i=1}^n \langle e_i, \lambda \rangle \phi_i.$$

It remains to apply the Stanley-Reisner description given by Theorem 6.4 proved in the previous subsection. \square

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